Stability Considerations for Synchronization of Multi-Dimensional Chaotic Flows using the Variable Replacement Method

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Sending sensitive information between remote locations can be accomplished using coupled chaotic oscillators to mask and retrieve the information. The variable replacement method of Pecora and Carroll provides a simple method to synchronize transmitting and receiving systems. This paper examines variable replacement within the Lorenz equations as a method for achieving synchronization of multi-dimensional flows. We discuss stability considerations imperative to system designers and show that Lyapunov exponents provide a ready litmus test to differentiate between synchronizing and uncoupled systems. To complete this survey, we study how synchronization depends on system parameters and initial conditions and present practical design considerations that motivate this analysis.

The ability to mask signals containing information has important applications in communications and signal processing. One possible method to safely transmit a message between remote locations involves chaotic oscillators producing multi-dimensional flows as a source of pseudo-randomness for masking information. A challenge in this setting is to obtain the same masking signal at both sending and receiving ends to allow successful encryption and decryption. This process of achieving an identical multi-dimensional masking signal at both sending and receiving systems given only a partial, lower-dimensional masking signal shared between both is called *synchronization*, and remains a hot topic of research in dynamic systems.

The synchronization process starts with the transmitting chaotic oscillator, which we call the "driver" because it produces the pseudo-random masking signal used for encryption. The driver emits both the masked message and a partial component of the masking signal, which travel to the receiving system. Here, the partial masking signal flows into the "response" oscillator governed by an identical system of equations as the driver. This input synchronizes the response oscillator, causing it to produce the complete original masking signal for decrypting the message. In practice, the governing equations behind each oscillator can be physically achieved as electrical circuits, where the equation parameters correspond to the values of circuit components (e.g. capacitors, resistors) [4].

Designing a communications system based on synchronized chaotic oscillators is a challenge for many reasons. First, the mathematics behind coupling chaotic oscillators are nontrivial. Additionally, practical concerns related to implementing equations in circuitry are numerous. Finally, achieving a protocol robust to perturbations is particularly important because long-term or sensitive installations need to recover from the introduction of noise into the system. Thus, considering all the possible design considerations involved is compulsory for successful system design.

In this paper we will survey the mathematical and practical design decisions required for inventing and implementing synchronized chaotic oscillators. First, we define and explore one method, variable replacement, that enables the synchronization of chaotic oscillators. Furthermore, we will conduct a case study using the Lorenz equations to demonstrate the process and functionality of synchronization. Later, we will discuss the role of Lyapunov exponents in characterizing system stability under perturbations. Finally, we will study the influence of parameter space and initial conditions on synchronization to assess concerns about practical implementation.

How Synchronization Works – A synchronized oscillator is derived from a system of dynamic equations exhibiting chaotic behavior. It must contain a drive system of continuous equations (which masks the transmitted information) coupled with a response system of continuous equations (which recovers the masked information.) The response system is identical to the drive system in form, but has a different set of operating variables. For example, where the drive system may have a vector $\vec{x}_D$ of state variables x, y, z, the response system has a vector $\vec{x}_R$ of corresponding state variables u, v, w.

The Lorenz equation provides an excellent case study for demonstrating synchronized oscillation. It is a three dimensional nonlinear dynamic system exhibiting chaotic behavior which is frequently used for weather and climate prediction. The governing drive equations are:

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y-x)\\ \frac{dy}{dt} &= x(r-z)-y\\ \frac{dz}{dt} &= xy-bz\end{aligned}$$

As mentioned above, the corresponding response equations below are identical to the drive equations, however they utilize the set of variables $\vec{x}_R$ instead of $\vec{x}_D$:

$$\frac{du}{dt} = \sigma(v-u)$$

$$\frac{dv}{dt} = u(r-w) - v$$
$$\frac{dw}{dt} = uv - bw$$

To demonstrate how to synchronize the equations we will use the replacement of variables method identified by Pecora and Carroll [2]. Synchronization is achieved by replacing one of the variables in the response equation with the corresponding variable from the drive equation. With the right replacement, the dynamics of the response system evolve such that all its state variables converge to the trajectory of the drive system even though only one drive variable is shared. Synchronization occurs when both system trajectories align: when $\vec{x}_D = \vec{x}_R$.

As an example of variable replacement that produces synchronization, we can replace the v in the top statement of the response equations with the corresponding y from the drive equation:

$$\frac{du}{dt} = \sigma(\mathbf{y} - u)$$

To visualize the trend the driver and response variables can be plotted against each other, as shown in figure 1.

Synchronization is evidenced by the fact that each of the modeled phase diagrams converge where $\vec{x}_D = \vec{x}_R$ for each independent variable. As mentioned above, the replacement process does not always produce synchronization. Only the exchange of certain variables in certain equations results in synchronization. For an example of a non-synchronizing replacement, consider replacing the v in the second line of the response equations with the y from the corresponding drive equation:

$$\frac{dv}{dt} = u(r-w) - \mathbf{y}$$

Plotting the drive and response variables in figure 2 shows that this replacement does not yield synchronized behavior.

Stability Analysis for Synchronization — When selecting a chaotic system to achieve synchronization objectives, determining the system's potential to synchronize is essential, as we saw that not all replacement schemes achieve synchronization. A system is stable for synchronization purposes if the response system can be perturbed and then resume its synchronized behavior naturally. Stated mathematically, we define a system aligning response variable $x_R(t)$ to the drive signal $x_D(t)$ be asymptotically stable (and thus synchronized) if it satisfies the condition

$$\lim_{t\to\infty} |x_R(t) - x_D(t)| = 0 \quad (1)$$

Given the same initial conditions to systems with identical governing equations and parameter sets, the two systems will be synchronized for all time. However, in practice it is important to ensure that if a small perturbation is introduced such that $x_R \neq x_D$, this error will die out exponentially over time. To determine this via mathematical analysis, we consider the conditional Lyapunov spectrum of the response system [1]. The Lyapunov spectrum of a system of equations contains a Lyapunov exponent for each dimension of the state space. We can think of a conditional Lyapunov exponent L as a factor governing the how the the error δx between drive and response variables evolves over time, as shown below

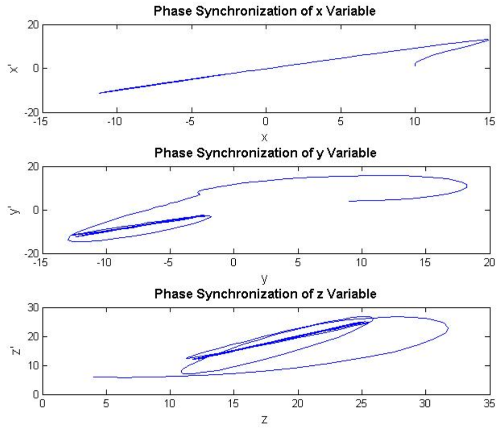


FIG. 1: Phase plots of the time evolution of synchronized Lorenz oscillators. Views of corresponding drive-response variable pairs x vs. u, y vs. v, and z vs. w are provided. All show convergent behavior along a line of unit slope after transient initial period. This convergence to a line of unit slope is the hallmark of synchronization.

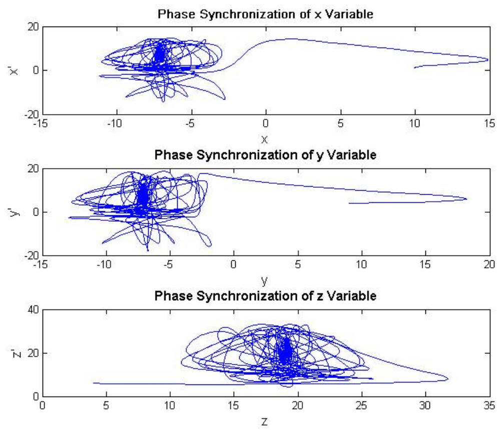


FIG. 2: Phase plots of the time evolution of Lorenz oscillators coupled by a variable replacement that does not produce synchronization. Views of corresponding drive-response variable pairs x vs. u, y vs. v, and z vs. w are provided. None show convergence to the unit slope, indicating that this particular variable replacement fails to produce synchronization.

in a simplified approximation

$$\delta x(t) = e^{Lt}\delta x(0) \qquad (2)$$

If the exponent is positive, the perturbation follows positive feedback to yield exponentially divergent behavior, while if the exponent is negative the error dies out over time to allow synchronization. Considering the entire spectrum, all exponents must be negative to produce synchronized behavior at a systematic level. Characterizing the conditional Lyapunov exponents of the governing equations of the response system thus serves as the litmus test for whether the system will provide asymptotic synchronization. We use the term *conditional* to emphasize that the value is determined for a certain dynamical state of the drive, so the exponents of a system may change under different conditions [1].

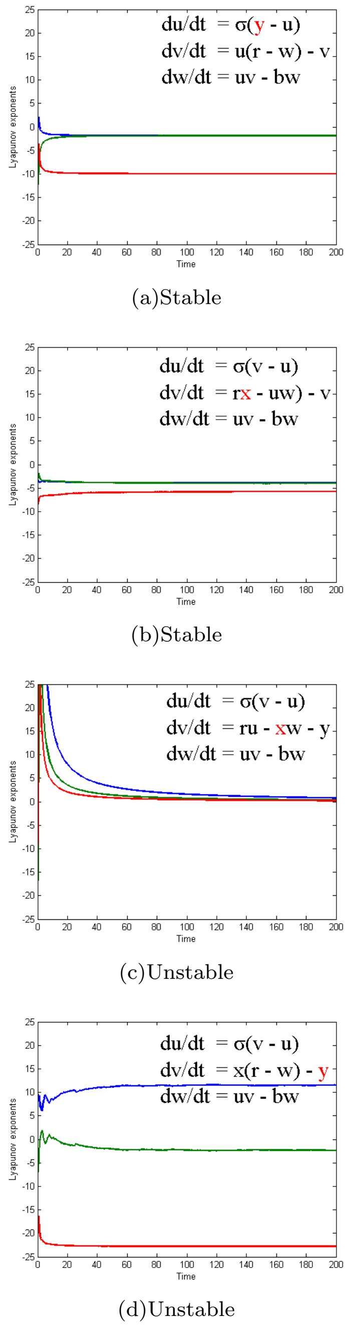
To observe this litmus test in action, compare and contrast the plots in figure **??**. Here we show plots of the numerically determined conditional Lyapunov exponents over time for several different possible response governing equations. Each possible equation represents a different way the drive and response systems could be coupled via the transmission of a drive state variable. Some choices (e.g. 3(a), 3(b)) show all three Lyapunov exponents converging to stable negative asymptotes, indicating their stability for synchronization. In contrast, other choices (3(c), 3(d)) have one or more positive asymptotic values, which yield positive feedback to perturbations and preclude synchronization.

To compute the exponents, we employed a numerical method derived from the open-source software published in [3]. At its core, the calculation of the Lyapunov exponents requires sophisticated continuous mathematics. In short, we need to track how small differences $\delta\vec{x}$ in trajectories through state space change over time and determine whether they converge or diverge. We compute this evolution over many small time periods. In each short interval, we linearize the governing equations and numerically solve the differential equation below to recover information about how small changes $\delta\vec{x}$ evolve over short bursts of time.

$$\frac{d(\delta\vec{x}(t))}{dt} = J|_{x(t)} \cdot \delta\vec{x}(t) \qquad (3)$$

where J indicates the Jacobian of the response system governing equations, evaluated at the current state $\vec{x}(t)$.

After resolving the time evolution in that interval, we can use Gramm-Schmidt orthonormalization to obtain a set of orthonormal vectors $\delta X_1 \ldots \delta X_d$ that span the differential trajectories. These vectors naturally orient along principal axes of growth or decay. After accumulating many of these sets to cover a long span of time, we can find the Lyapunov exponents via a time average, shown below



(a)Stable

(b)Stable

(c)Unstable

(d)Unstable

FIG. 3: Side-by-side comparison of Lyapunov exponentials for three versions of the response governing equations. Systems with all exponents converging to negative asymptotic values allow stable synchronization, while systems with any positive asymptotic exponents do not.

$$\lambda_i = \frac{1}{N \cdot T}\sum_{k=1}^{N} \ln ||\delta X_i^k||, i = 1, 2, \ldots d \qquad (4)$$

A more detailed explanation of the Lyapunov exponent calculation process is beyond the scope of this work. Refer to Chapter 2 of [1] for a thorough treatment of the mathematics and numerical considerations involved.

Synchronization, Parameter Space, and Initial Conditions — One final question of interest to this introductory tour of synchronized chaos is the influence of the parameter space and initial conditions on the stability of a system designed for synchronization. From a practical perspective, finding a drive-response system that is robust to initial conditions and parameters is advantageous for two reasons. First, a system too sensitive to state space might be thrown off by perturbations and be unable to recover. Second, a wide space of parameters and initial conditions allows the developer to optimize other constraints. For example in circuit design, research implementations often require particular parameter choices to meet the limitations of available electronics, as mentioned in [4].

We chose to study the drive-response indicated in figure 3(a) where $\frac{du}{dt}$ is connected to the drive signal y. In order to determine its robustness for synchronization, we evaluated its conditional Lyapunov spectrum across a wide range of the Lorenz parameter $r \in [20, 200]$, a region known to yield chaotic behavior. For each iterated trajectory, we chose a random initial condition in the state space of both drive and response systems such that x, y, z, u, v, w were selected from uniform distributions across the interval $[0, 1]$. We evolved each trajectory for 200 time units and recovered the asymptotic limit for each Lyapunov exponent. The two largest exponents are plotted across r in figure 4. It appears that these exponents are consistently confined within the intervals $(-1.75, -1.85)$ and $(-1.85, 1.95)$, respectively. The third exponent (not shown) is similarly restricted around -10. This plot provides evidence that this system appears stable across the surveyed space because its entire spectrum satisfies the less-than-zero litmus test. It also independently confirms a similar plot of exponents for various values of r in [1], verifying our numerical implementation. Furthermore, this stability appears relatively independent of parameter r and initial conditions, indicating an underlying structural stability. This robustness is highly desirable from a system designer's perspective for reasons articulated above.

Conclusion — Sending sensitive information securely between remote locations can be possible via the synchronization of a driver and receiver system of equations. In this paper we provided insight into the systemic parameters for selecting an appropriate system of equations with the Lyapunov criteria, and provided a guide for synchronizing the appropriate coupled drive and response systems. While the Lorenz equations appear to be robust across parameter space and initial conditions for this purpose, synchronization has applications in the physical world and there may be other systems of equations which are better suited for different applications. Furthermore, the direct replacement of terms in the response system to establish a link to the drive is only one possible method to achieve synchronization. Others are certainly possible, such as a continuous control scheme [1].

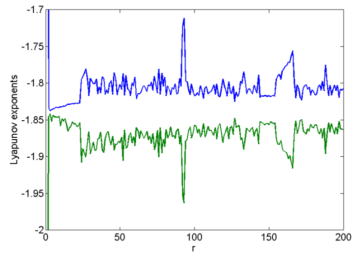


FIG. 4: The asymptotic values of the two largest conditional Lyapunov exponents computed across the range of Lorenz parameter r for response system shown in 3(a). Observe that the exponent values are consistently negative across the parameter space, yielding reliable synchronization behavior that seems independent of chosen parameters. Exponents calculated with parameters $\sigma = 10, b = 8/3$ over the time interval $t \in [0, 200]$. The third value hovered consistently around -10.

As such we recommend that someone interested in transmitting sensitive information using this method of synchronization identify their needs to select the most appropriate combination of equations. Different systems of equations and methods of synchronization emphasize different qualities and aspects of the transmission. Possible design considerations include: the amount of power required for transmission, the accuracy of transmission and retrieval, and sensitivity to external perturbations. The selection of driver and receiver systems should reflect these needs as well as the less-than-zero litmus test for Lyapunov spectra we have discussed in this paper to provide the most appropriate implementation for the specific application.

[1] J. M. Gonzalez-Miranda. Synchronization and Control of Chaos: An Introduction for Scientists and Engineers. Imperial College Press, 2004.

[2] L. M. Pecora and T. L. Carroll. Synchronization in Chaotic Systems. Phsyical Review Letters **64** 8 (1990).

[3] Vasiliy Govorukhin. "Calculation Lyapunov Exponents for ODE". Open source Matlab code available online. `http://www.mathworks.com/matlabcentral/fileexchange/4628-calculation-lyapunov-exponents-for-ode`.

[4] K. M. Cuomo and A. V. Oppenheim. Circuit Implementation of Synchronized Chaos with Applications to Communications. Physical Review Letters **71** 1 (1993).