Numerical Valuation of Stock Options using the Black-Scholes PDE

Mary Germino and Michael C. Hughes, Class of 2010
Olin College of Engineering

Abstract

The Black-Scholes partial differential equation assists financial analysts in assigning fair values to stock options based upon current and projected market trends. In this paper, we discuss basic option valuation theory, provide qualitative motivation for the Black-Scholes model, and present a numerical approach using the FTCS technique to evaluate European-style options. We then offer some analysis of the convergence behavior of this technique under different mesh sizes and provide preliminary error analysis comparing this technique and the closed-form analytical solution. The concepts applied to the development, implementation, and analysis of our numerical model apply to a wide range of other problems that an undergraduate student of mathematics or physics might face.

I. Introduction

Numerical solutions to partial differential equations, as taught to undergraduate students of mathematics and physics, are of great significance in the evolving field of economics. A particular application in this field, which will be the focus of this paper, is the pricing of stock options using a numerical solution to the Black-Scholes equation.

In finance, a stock option involves a contractual transaction between the option's *holder* and the option's *writer*. When purchasing a stock option, the holder gains the right (but not the obligation) to at some later date either acquire from the writer or sell to the writer some stock asset for an agreed upon price. This price is known as the *exercise* or *strike price*. When the holder has the option to buy the stock from the writer, the option is called a *call* option. When the option involves the holder selling the stock to the writer, it is termed a *put* option. If the holder can only exercise the option at one specific time in the future (called the *expiry time*) the option is considered to be *European*. Alternatively, an option that the holder can exercise at any point from the time of purchase to the expiry time is termed *American*.

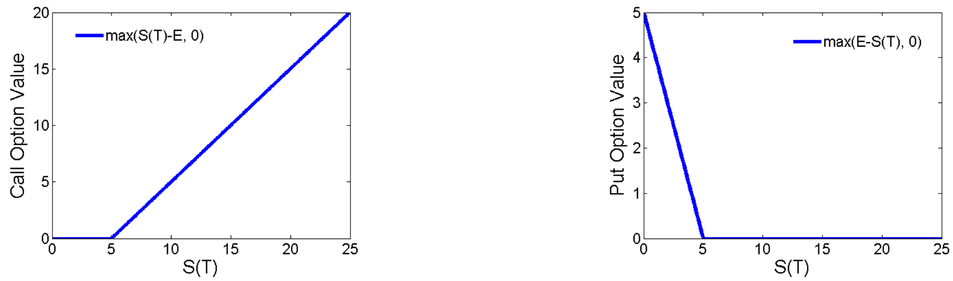
In this work, we are concerned with determining how to fairly determine the purchase price of a stock option, such that the writer of the option has, at least in theory, the same chance of making or losing money on the deal that the holder does. Without a fair pricing, either the writer or the holder has no reason to enter into the contract. Work done on this problem in 1973 by Robert Merton, Fischer Black, and Myron Scholes eventually resulted in the celebrated Black-Scholes equation and a 1997 Nobel Prize in Economic Sciences for Scholes and Merton. The Nobel Prize press release in 1997 lauds their work as "among the foremost contributions to economic sciences over the last 25 years" [3]. The Black-Scholes model provides financial analysts a basis for fairly pricing stock options and allows savvy investors a means to capitalize on predicted future gains or hedge against expected future losses. The model can be applied to a variety of option styles simply by altering the boundary conditions involved, making it usable with European, American, and most exotic styles of options trading.

Closed-form analytical solutions to this equation exist for the evaluation of European-style options, but no such analytical solution exists for American-style options, which are more widely traded due to the ability to exercise at any time. Thus, we focus on applying numerical techniques to solving the Black-Scholes equation. We chose to specifically examine the use of FTCS methods in solving the equation. Additionally, it is a useful exercise to numerically solve the Black-Scholes equation even for European options, since this allows for comparison to analytical solutions and can later be expanded for application to other options. We thus choose to limit our work to the domain of European-style options.

First, we discuss in more detail the mathematical theory behind the values of different styles of options. Next, we motivate the Black-Scholes model of stock option valuation and discuss the Black-Scholes PDE itself. Later, we derive a discrete FTCS approximation of that PDE for use in numerical modeling, and then offer some analysis of the convergence behavior of this model under different mesh sizes. Finally, we compare the simulated solution to the analytical solution and conduct preliminary error analysis.

II. Basic Option Valuation Theory

The choice to exercise a European-style option at its expiry time $t = T$ is fundamentally based on the potential profit the holder expects to gain from buying or selling the underlying asset at that time. The expiry value of the option can be considered as a function of the agreed upon strike price E and the price $S(T)$ of the stock asset in the market at that time. Knowing the basic mathematical structure of this function allows us to properly understand the underlying boundary conditions required to solve the Black-Scholes equation discretely.



(a) Call Option Value

(b) Put Option Value

Fig. 1. European option value at expiry of a call (a) and a put (b) as a function of stock price with strike price E = 5. The call option only has value when S(T) is greater than E, while the put option only has value when S(T) is less than E. The overall hockey stick shape is characteristic of option valuation, with the bend in the stick being the point at which the option transitions from having no value to having value.

In the case of a European call option, the holder wishes to exercise the option and buy the stock at the strike price only if the current market value of the stock at expiry is greater than the strike price. If he does so, he can turn around immediately and sell the stock for the current market price, making a profit of $S(T) - E$. If the stock is trading at some value less than or equal to the strike price, the holder would simply choose not to exercise the option and buy the stock at the lower market price if so inclined, rendering the option worthless. We thus can express the value $V_{call}(T)$ of a call option at expiry time T to be

$$V_{call}(T) = max(S(T) - E, 0) \tag{1}$$

A graph of call option value as a function of the underlying stock price can be seen in Figure 1(a). The ”hockey stick” shape of this graph is characteristic of option valuation, with the sharp corner occurring where $S(T) = E$ and the option transitions from being worthless to having value.

Similarly, a European put option is only worth something to the holder if the current stock price is lower than the strike price, such that exercising the option and selling the stock at the strike price would net the holder a profit of $E - S(T)$ when compared to simply selling the stock at market value. Otherwise, if the strike price is at or below the current stock price, it becomes worthless to the holder. We express the value $V_{put}(T)$ of a put option exercised at expiry time T as

$$V_{put}(T) = max(E - S(T), 0) \tag{2}$$

A graph of put option value as a function of the underlying stock price can be seen in Figure 1(b).

We can thus easily compute the value of a European-style stock option at expiry when we know the stock price at that time. However, the process of predicting the expiry value of a stock option *before* expiration knowing only current market values is much more difficult. The Black-Scholes model was derived in an attempt to provide a fair, rational basis for this prediction of an option's future value.

III. The Black-Scholes Model

A. Model Assumptions

There are several assumptions inherent in the Black-Scholes model of stock option valuation. First and foremost, the stock price is assumed to follow a geometric Brownian motion with constant volatility. Additionally, the model assumes no possibility of price discrepancies between the markets involved (i.e. all markets are arbitrage-free). Trading of the options is assumed to take place in continuous time and allows stock value to be divisible into arbitrary fractions. The model assumes it is possible to short sell the stock involved as well as borrow and lend cash at a constant risk-free interest rate. Finally, transaction fees, dividends, and taxes associated with the options are not taken into consideration. [1,2]

B. The Black-Scholes PDE

The complete mathematical derivation of the Black-Scholes equation is well beyond the scope of this work, but an excellent account is given by Chapter 8 of Higham [1]. To briefly motivate the model, let us consider an investment portfolio consisting of some actual stock assets trading at predicted market prices and some cash assets stored in a bank account accruing risk-free interest continually at rate r. The components of this portfolio are continually rebalanced such that risk of the portfolio always mirrors the risk of the option in question. Because of the no-arbitrage assumption and the assumption of constant, risk-free interest rate, the rate of return on this portfolio should equal the rate of return on the option at all times.

Thus, by deriving an expression for the value of the portfolio over time and using a Taylor series to approximate the value of the option over time, we can equate their rates of return and find an expression for the value of the option. From this we arrive at the Black-Scholes PDE, which relates an option's value V to the underlying asset price S and time to expiry τ as shown below.

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{3}$$

We note that r represents the constant, risk-free interest rate and stock price S is expected to vary according to Brownian motion with a constant volatility σ.

IV. Model Approximation of Black-Scholes

A. Discretization

As mentioned previously, a closed-form analytical solution to Black-Scholes does not exist for American-style options and other more exotic options. Therefore, finding a solution to the function $V(\tau, S)$ often requires using discrete approximation techniques. In our European-style case (where an analytical solution exists), applying numerical techniques can still prove useful in that it provides a means to validate our method.

For its low barriers to understanding and implementation, we have chosen to implement an FTCS (Forward difference in Time, Central difference in Space) numerical technique. We discretize the Black-Scholes equation (3) by replacing all derivatives with FTCS finite difference approximations as follows

$$\frac{V_j^{i+1} - V_j^i}{k} - \frac{1}{2}\sigma^2 S^2 \frac{V_{j+1}^i - 2V_j^i + V_{j-1}^i}{h^2} - rS\frac{V_{j+1}^i - V_{j-1}^i}{2h} + rV_j^i = 0 \tag{4}$$

where V_j^i represents the option value grid point with time index i and stock price index j. Time is given as $\tau = ik$, and stock price value is given as $S = jh$ where k is time step size and h is stock price step size. We note that the discrete approximation tracks time t, not time until expiry $\tau = T - t$, which causes the sign changes between equations (3) and (4).

By solving the above equation for V_j^{i+1}, we transform it into a form solvable with FTCS.

B. Boundary Conditions

With can use our basic knowledge of option valuation to discover the proper boundary conditions for both call and put options. For the sake of brevity, we will only derive the call option boundary conditions here. The put conditions can be obtained with similar processes.

In the case of call options, we recognize that if the stock price drops to zero, the stock would be taken off the market and the option would be worthless. We thus have for all t the condition:

$$V(t, 0) = 0 \tag{5}$$

For the second boundary condition we realize that as the stock price approaches infinity, the call value also approaches infinity. For the purposes of simulation, and because stock prices never truly go to infinity, it is sufficient to choose some relatively large number L to be the cap on stock prices, and consequently, option values. Thus, we have

$$V(t, L) = L \tag{6}$$

Finally, we know that, at expiry, a call option's value follows the "hockey stick" function shown in Figure 1(a),

$$V(T, S) = max(S(T) - E, 0) \tag{7}$$

C. Implementation

To implement this discrete approximation, we first select the number of grid points for stock price N_s and time N_t. We then define the stock price step size h and the time step size k as

$$h = \frac{L}{N_s} \quad k = \frac{T}{N_t} \tag{8}$$

For our simulation, we maintain an expiration time T of 1, a stock price cap L anywhere from 10 and 50, and an exercise price from $\frac{1}{3}L$ to $\frac{1}{2}L$. We use stock volatility values $\sigma \approx .5$ and interest rates $r \approx .05$. We did not adopt particular units for our times and prices, increasing versatility and portability among different markets. A plot of our Matlab implentation of a discrete approximation for the Black-Scholes equation for European call options is given in Figure 2.

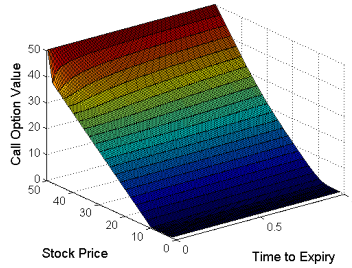


Fig. 2. Discrete solution of the Black-Scholes equation showing fair-price European call option value as a function of stock price and time until expiry. We see that the boundary case of expiry time ($\tau = 0$) assumes the characteristic hockey stick shape discussed in Figure 1. As the graph moves away from this boundary, the hockey stick shape becomes increasingly smooth and rounded. Additionally, we note that the most noticeable difference between discrete and analytical solutions accords at the highest values of S, since the approximation of L for infinity is inaccurate.

V. Model Analysis

A. Convergence Analysis

Vital to the success of the discrete solution to the Black-Scholes equations is the choice of appropriate mesh sizes such that the FTCS approximation actually converges to the solution.

Rigorous mathematical investigation which is beyond the scope of this work (see Chapter 24 of Higham [1]) shows that for any implementation of FTCS to be convergent in the sense of von Neumann, one must choose appropriate grid step sizes h and k such that the ratio $\nu = k/h^2$ satisfies

$$\nu \leq \frac{1}{2} \tag{9}$$

However, the bound of $\frac{1}{2}$ is by no means the least upper bound for ν in all applications of FTCS. For our particular implementation, we find that the upper bound was in fact much lower. We thus wish to discover exactly where the boundary between convergence and failure for our discrete FTCS simulation lies, so that we can properly choose mesh sizes and spacings to achieve meaningful results. Ideally, we'd like to determine the relation between N_s and N_t at this stability boundary. However, preliminary efforts to find this relationship were unsuccessful. We do know that it is dependant on at least expiration time T and stock price cap L.

Regardless of the choice of values for T and L, the general shape of the stability curve for our model follows that given in Figure 3.

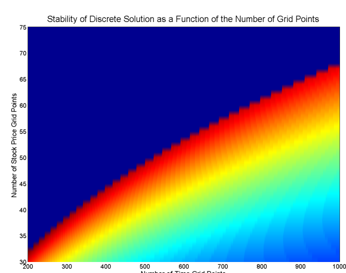


Fig. 3. Convergence stability of the FTCS discrete solution to the Black-Scholes equation as a function of the number of grid points for stock price N_s and time N_t. The dark blue region represents unstable grid value combinations which do not converge. The gradient of colors from red to light blue represents stable convergent combinations. The coloring of this gradient represents the ratio $\nu = k/h^2$, where k is the space between time points and h is the space between stock price points. We see that there seems to be some limiting function which delineates stable combinations of N_t and N_s from unstable ones. Additionally, we note that the ν values at the border seem to be constant, since they are all approximately the same color.

B. Error Analysis

For European options, an analytical solution to the Black-Scholes equation is possible, if rather unwieldy. Because of its complexity, we omit the presentation of such a solution, and refer the reader to Chapter 8 of Higham [1]. We compared our model's performance to the analytical solution to determine the error of our numerical solution. A plot of the maximum error of our model as a function of the ratio between strike price E and the imposed cap on stock price L is shown in Figure 4. Interestingly, the maximum net error between discrete and analytical solutions increases linearly with increasing $\frac{E}{L}$. If we look back at Figure 2, we see that this maximum net error likely results from the artificial maximum boundary L imposed on stock price. As we increase the value of L, or the relative space between E and L, the value of $\frac{E}{L}$ decreases and the discrete approximation becomes closer and closer to the analytical solution.

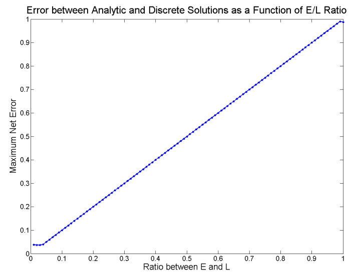


Fig. 4. Maximum net error between analytic and discrete solutions as a function of the ratio of strike price E and limiting stock price L. Maximum net error was computed via the greatest net difference between corresponding grid point solutions using analytic and discrete methods. We see a clearly one-to-one linear relationship between this error and the ratio between E and L, indicating that for a given strike price, one can easily compute the liming price to use in the discrete solution process to yield a value within a desired error tolerance. Evidence of leveling behavior is visible at extremely low E to L ratios, revealing that there may exist some mandatory minimal error which is independent of E or L.

Increasing the number of grid points does not affect the maximum error, which is determined solely by the ratio of exercise price to the stock price cap. Regardless of the size or resolution of the grid, there will always be nodes at $S = L$, at which occur the greatest difference between the discrete approximation and the analytical solution.

VI. Conclusion

The Black-Scholes partial differential equation is an excellent application of the numerical solving techniques studied by undergraduate students of mathematics and physics. Here, we summarized the process for creating and analyzing a model of the solution to the Black-Scholes equation for European call options. We examined the convergence behavior of the model and computed the error between our discrete approximation and the analytical solution. We determined that this error has a direct linear relationship with the exercise price to stock price cap ratio. Our work follows the many assumptions inherent in the Black-Scholes formulation and finds much more utility as an instructive exercise in applied numerical modeling of partial differential equations than as an actual contribution to the field of stock option valuation.

A fitting continuation of this project could include the modeling of any of the remaining types of options, which would likely present a new spectrum of simulation challenges. In particular, we would like to create a model for American stock options, which are options that may be exercised any date up to and including the expiration date. This would require a model very much like the one made for the European options, except with a ”moving” boundary condition. Additional expansions could involve valuing barrier options, combination options, or other, more exotic option styles.

Acknowledgment

The authors would like to thank Dr. John Geddes and Dr. Mark Somerville for their assistance and direction throughout this project. We also thank David Wurtz (Olin ’07) for providing valuable research materials.

References

[1] Higham, Desmond J., *An Introduction to Financial Option Valuation* - Mathematics, Stochastics, and Computation. Cambridge, UK: Cambridge University Press, 2004.
[2] ”BlackScholes.” Wikipedia, The Free Encyclopedia. 4 May 2007, 17:02 UTC. Wikipedia Foundation, Inc. 7 May 2007 `<http://en.wikipedia.org/w/index.php?title=Black\%E2\%80\%93Scholes&oldid=128243899>`.
[3] The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1997, Press Release. 14 October 1997. `<http://nobelprize.org/nobel_prizes/economics/laureates/1997/press.html>`.