On Pure and Approximate Nash Equilibria in Betweenness Centrality Games

Xiaohui Bei*, Wei Chen †, Gunes Ercal-Ozkaya‡, Xiao Fu§, Michael Hughes¶ and Sarah McBride‖

*Tsinghua University, beixiaohui@gmail.com

†Microsoft Research Asia, weic@microsoft.com

‡University of Kansas, gunes.ercal@gmail.com

§Salem College, fuxiao718@gmail.com

¶Franklin W. Olin College of Engineering, michael.hughes@students.olin.edu

‖Brigham Young University, sarah.mcbride925@gmail.com

Abstract

In this paper, we study a network formation game in which each node in the network attempts to maximize its betweenness centrality, a game originally proposed by Chen et al. [4]. We prove connectivity properties of pure Nash equilibria for both directed and undirected versions of the game. We conduct extensive simulations to explore various properties of stable graphs, such as connectivity, symmetry, and fairness. We also study types of graphs that provide good approximation ratios to pure Nash equilibria. Our results provide better understanding of the stable structures that can arise when selfish agents each attempt to maximize their own share of traffic flow in a social or computer network.

I. Introduction

One currently active area of theoretical computer science seeks to understand how selfishly-motivated agents collectively build a network to access shared resources. Investigating theoretical network formation contributes to our understanding of real-world decentralized network phenomena such as social networks, peer-to-peer networks, and the Internet, which has been considered "the most important computational artifact of our time" [9]. A convenient and effective way to model selfish network formation is to consider the process as a game, in which each member of the network is a player trying to maximize its benefit from the network. Using this approach, researchers can leverage the mathematical tools of graph theory and game theory to analyze network formation [9]. Many researchers have utilized this approach to examine the Nash equilibria *stability* of autonomously-created networks [1], [13], and [7].

Players in different network formation games may have different optimization objectives. For example, a peer may want to minimize its average distance to all other peers, a metric known as *closeness centrality*. There are several papers studying network formation games related to closeness centrality, such as [1], [13], and [7]. Alternatively,

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another metric exists known as *betweenness centrality*. A peer's betweenness centrality indicates the amount of traffic which passes through it within the network, assuming shortest path routing is used in all cases. More formally, for a graph G with vertex set V of size n and edge set E, we define the betweenness centrality $B(v)$ for any vertex $v \in V$ as

$$B(v) = \sum_{s \neq t \neq v \in V} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

where σ_{st} is the total number of shortest paths from s to t, and $\sigma_{st}(v)$ is the number of shortest paths from s to t that pass through vertex v.

In many real-world scenarios for decentralized network formation, maximizing betweenness centrality may be a natural objective for an agent. Consider a computer or peer-to-peer network in which a node can charge its peers for the amount of traffic it helps deliver throughout the network. In this case increasing a node's revenue is directly dependent on improving its betweenness centrality. Alternatively, in many social networks it may be beneficial for a member to be a highly-used conduit for information. This allows the agent both control over distribution and an opportunity to be one of the first in the know. Social actors, then, may have an incentive to improve their value in the network by increasing betweenness centrality.

Recently, Chen et al. propose to study network formation games in which each node (or peer) in the network attempts to maximize its own betweenness centrality while constrained by a limited budget to build connections in the network [4]. This limited budget attempts to reflect real-world constraints such as costs for nodes in computer networks and finitely available time and energy for social actors. Chen et al. call these games *bounded budget betweenness centrality* games, or B^3C games. They study both a general version of the game where nodes have limited budgets and links have different costs, and a uniform version of the game where each node can build k links to other nodes. They show a few results such as in the general version determining if Nash equilibiria exist is NP-complete, but very few positive results on the structure of Nash equilbiria are presented.

In this paper, we extend the work in [4] in a few directions. First, we offer a polynomial algorithm to find an approximation of the best response of any player in the B^3C game, even though [4] showed that determining the exact response is NP-Hard under certain conditions. Second, we include undirected graphs in the study, while in [4] only directed graphs are considered. Third, we focus more on the structural properties of Nash equilibria in uniform games. In particular we show that undirected Nash equilibria must be connected and directed Nash equilibria must be weakly connected. We use simulations to search for Nash equilibria and reveal that all small-sized directed Nash equilibria found thus far are strongly connected. Furthermore, many of these stable graphs are Eulerian, contain a Hamiltonian cycle or path, and have other symmetric properties. Finally, we study approximate Nash equilibria and investigate how close certain graph structures are to pure Nash equilibria as well as how various properties correlate with stability. We close by suggesting directions for continued work in this area and list many conjectures about the B^3C game for further investigation.

These studies increase our knowledge of the uniform B^3C game model, motivate future research, and lend insight on the stable structures which can result from real-world network formation in which maximizing traffic flow through an agent or node is the primary member objective.

II. Heuristic in the Computation of Best Responses

The *best response* of a player in the B^3C game is the strategy which yields the greatest betweenness centrality score possible given a set of fixed strategies of other players in the network. For a given graph $G = (V, E)$, the best response of a particular node $v \in V$ is the k-sized set of other nodes $W = \{w_i \in V \setminus \{v\} \mid i = 1, 2, \ldots, k\}$

such that, when v connects (builds an edge) simultaneously to each member of W, the betweenness centrality score $B(v)$ for node v achieves its greatest possible value for the fixed configuration of other nodes in G.

Computing the best reponse of a node given the strategies of other nodes is a difficult task. In fact, as shown in [4], it is NP-hard to compute the best response for any node in the graph if both n and k are variables. Even if k is a constant, the time complexity is still in $O(n^k \text{poly}(n))$. To circumvent this issue, we use the following single-edge contribution heuristic to achieve an much faster approximation for the best response.

Let $B_w(v)$ denote the betweenness of v in graph G if is only connected to one other node w. Let $B_{w_1,w_2,\cdots w_k}(v)$ denote the betweenness of v in graph G if is connected to all nodes in the set $\{w_1, w_2, \cdots w_k\}$. We bound $B_{w_1,w_2,\cdots w_k}(v)$ using $B_{w_i}(v)$'s, as shown in the following lemma.

Lemma 2.1: The betweenness of a node is bounded from above by the following inequality:

$$B_{w_1,w_2,\cdots w_k}(v) \leq \sum_{i=1}^{k} B_{w_i}(v).$$

This lemma allows a significant speed-up to the typical case of determining the best response of a single node v. We can first determine the response of v due to each possible single-edge partner, and then we sort all k-size sets of possible partners in descending order by the sum of their single-edge contributions. We then only need to test k-size groupings from this ranked list until we find one whose sum falls below the best response for v computed so far. Thus, we do not always need to test all k-size groupings to find an upper bound on any node's betweenness.

Lemma 2.1 also suggests a base-line approximation of the best responses. Let $B^1(v)$ denote the best response v that can be achieved by connecting to one single partner, and let $B^k(v)$ denote the best response of v when allowed k partners. Then we have the following lemma:

Lemma 2.2: For any k, we have

$$\frac{1}{k} B^k(v) \leq B^1(v)$$

Proof Idea: The proof follows directly from Lemma 2.1, since $\sum_{i=1}^{k} B_{w_i}(v) \leq k \cdot B^1(v)$ for all w_i.

With this fact, we can get a $1/k$ approximation of the best response for a node as long as we include its best single response edge as part of our strategy. In this way we only need to perform the betweenness calculation on n different graphs to determine some node's best response. Thus, our total run-time complexity is $O(n^2 k)$ for sparse graphs or $O(n^4)$ for dense graphs (see algorithms given in [3]). So we can achieve a polynomial time $1/k$ approximation for the best response of any node v in the network.

III. Properties for Pure Nash Equilibria

In this section, we aim at studying different properties of pure Nash equilibria in the B^3C network formation games both analytically and via simulations.

A. Connectivity for Directed and Undirected Nash Equilibria

We show that all Nash equilibria in the undirected version of the game are connected, and all Nash in the directed graph game must be weakly connected. In the undirected game, we allow bidirectional traffic along each edge. Given this premise, we can show that any stable graph must necessarily be a connected graph. For full proofs, we refer the reader to Appendix A

Theorem 3.1: For the undirected version of the B^3C game, all Nash Equilibria are connected.

Proof Idea: We prove this by contradiction. Suppose there exists an undirected graph G which is at Nash Equilibrium but not connected. Then G has at least two internally connected but collectively disjoint components

A and B. Without loss of generality, we can suppose that $|A| \leq |B|$. We then establish two facts: A always has at least one edge e which can be removed without disconnecting A, and a vertex a incident to e can always obtain a higher betweenness score by removing edge e and instead connecting to any node in B. This implies v is not at a best response, so G cannot be stable. Thus, the only stable graphs are those which are connected. ■

In the directed game, we allow only one-way traffic along each directed edge. Stable graphs in this version of the game must be weakly connected. We term a directed graph *weakly connected* if its underlying undirected graph is connected. That is, if we consider every edge as bidirectional, the graph is connected.

Theorem 3.2: For the directed version of the B^3C game, all Nash Equilibria are weakly connected.

Proof Idea: We prove this by contradiction. Suppose there exists a directed graph G which is at Nash Equilibrium but not weakly connected. We can group the nodes of G into strongly-connected components and consider these as nodes in a new graph S. If G is not connected, then S is not connected and can be topologically sorted into at least two directed acyclic sub-graphs, C_1 and C_2. If we consider the respective source components S_1 and S_2 of C_1 and C_2, at least one node in S_1 is not at best response and would achieve a higher betweenness score by removing its edge and connecting to a node in S_2. This implies that this node is not at best response so G cannot be at Nash Equilibrium. Thus, directed graphs that are stable must be weakly connected. ■

B. *Simulation Results for Pure Nash Equilibria*

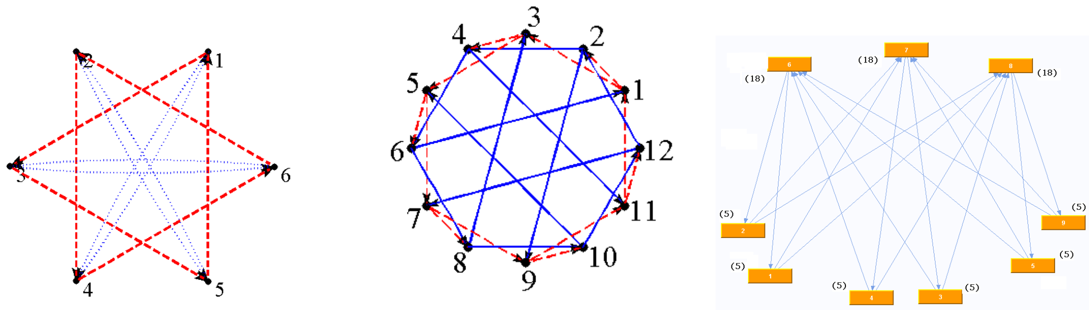
We used various methods to search for Nash equilibrium graphs with $k = 2$ outgoing edges per node as exhaustive search became infeasible for $n > 8$. As we generated more Nash equilibria, the properties of the observed Nash equilibria helped us to hone our search space for successively larger n in testing those properties or discovering new properties. Properties of interest for our search included connectivity, Eulerian degree, Hamiltonian paths, and fairness for each network. We define the fairness of a graph as the ratio between the minimal and maximal betweenness values found among its vertices.

A nice stable structure that arose repeatedly for small n in our simulations is the class of *Abelian Cayley* graphs.

Definition Let G be a group and S a subset of G (called the *generating set*) that does not contain the identity. The $Cayley$ graph is the directed graph with each node associated with one group element and edges described by the set $E(G) = \{(g, g \cdot s) | g \in G, s \in S\}$. An **Abelian Cayley Graph** is a Cayley graph where the associated group is abelian, meaning the group operation is commutative.

An example of an Abelian Cayley stable graph for the directed version of the game can be found in figure 1 (with corresponding generating set and betweenness indicated in the caption). Inspection of Abelian Cayley graphs reveals that they all have some interesting properties: strong connectivity, uniform betweenness (and thus perfect fairness) and equal in- and out- degree (which makes them Eulerian). The uniform betweenness occurs because Abelian Cayley graphs are vertex transitive. Finally, for all graphs with either a or b equal to 1 (and those isomorphic to such graphs) there exists a Hamiltonian Cycle, which was the case in many of the Nash equilibrium Abelian Cayley graphs that arose. As such, our isomorphism results have proven useful in helping to hone the equilibrium search. We refer the reader to appendix B for an explanation of how Abelian Cayley graphs are constructed, additional examples, isomorphism results and proofs, and further details.

Abelian Cayley graphs also arose as Nash equilibria for the *undirected* version of the uniform betweenness centrality game of $k = 2$ outgoing edges. Note that the way to determine betweenness of the node x in undirected graphs is different from that of directed graphs: To calculate x's betweenness, we count all the shortest paths which



(a) Abelian Cayley Generating Set: {2,3},B=4,4,4,4,4,4. (b) Partitioned Abelian Cayley, Generating Sets: $\{1, 2\}$, $\{2, 7\}$ (c) Asymmetric, $n = 9$

Fig. 1. Three graphs in Nash equilibrium: Abelian Cayley, Paritioned Abelian Cayley, and Asymmetric.

go through x, regardless of whether x built any of the edges in the path. We have identified stable Abelian Cayley graphs of the undirected game for $n = 4$ through $n = 15$ nodes. We have also found two stable Abelian Cayley graphs with $n = 18$. A picture of a stable Abelian Cayley undirected graph for $n = 18$, with generating set $\{1, 7\}$ is shown in Figure 12 in Appendix C.

For the directed betweenness centrality game, our simulations produced Abelian Cayley Nash equilibria up to $n = 11$, but none for larger n though we searched up to $n = 50$. This is consistent with the result in [4], which proves that for the directed $k = 2$ game, for sufficiently large n Abelian Cayley graphs cannot be Nash equilibria, but [4] does not give the actual upper bound on n where Abelian Cayley Nash can be found. Similarly, for the undirected version of the game, our exhaustive search of $n = 16, 17$, and $19 - 50$ did not yield any Nash equilibria. Therefore, we conjecture (like in the directed case) that Abelian Cayley undirected stable graphs *do not* exist for large n. Undirected Nash equilibria and their properties thus remain the subject of future research.

As we are primarily interested in properties that are indicative of Nash equilibria, we note other forms of equilibria obtained by our simulations and to compare and contrast corresponding properties: partitioned Abelian Cayley graphs and Asymmetric equilibria. Partitioned Abelian Cayley graphs, which we introduce here, have multiple generating sets, as opposed to one generating set like in the normal Abelian Cayley case. This results in every other node, or every third node, etc. to be locally similar. Note that the number of generating sets for each graph is a constant. In contrast, our searching also yielded Nash equilibria that did not exhibit any symmetry, which we term Asymmetric Nash graphs. Examples of all three types of stable graphs can be found in figure 1 along with explanations for the construction of Partitioned and non-Partitioned Abelian Cayley Graphs in Appendix B. While in figure 1 two generating sets were used to construct the partitioned Abelian Cayley graph, there also exist partitioned Abelian Cayley Nash equilibria with three or four generating sets found in Appendix B and in [4], respectively.

We note some immediate observations about both partitioned Abelian Cayley graphs and Asymmetric graphs that contrast with properties of Abelian Cayley graphs are as follows. In-degree does not always equal the out-degree. This may result in non-uniform betweenness and poor fairness as exhibited both by the partitioned Abelian Cayley graph of Figure 11 and the Asymmetric Nash graph of Figure 1 with fairness $\frac{5}{18}$. Furthermore, unlike all Abelian Cayley Nash equilibria discovered, both figures as well as the partitioned Abelian Cayley graph of Figure 1 indicate that such non-Abelian Cayley Nash graphs need not have a Hamiltonian path. However, a characteristic that has held true amongst all directed Nash equilibria found thus far is strong-connectivity.

IV. Directed Approximate Nash Equilibria

Examining approximate Nash when finding completely stable networks becomes intractable can be a practical alternative in network formation games research [5]. Approximate stability is also interesting in its own right: finding networks which have some fixed bound on their proximity to pure equilibria as graph size becomes arbitrarily large poses an interesting challenge. Moreover, properties of graphs which are almost stable may yield insights into properties of completely stable graphs.

We define the *approximation ratio* as a measure of proximity to stability for the B^{3}C game as follows. For any node i in a graph G, we define the ratio α_i as

$$\alpha_i = \frac{B(i)}{B_{best}(i)} \tag{1}$$

where $B(i)$ denotes the betweenness centrality of i within G and $B_{best}(i)$ denotes the best response of i given the fixed arrangement of other players in G. For the entire graph G, the approximation ratio $\alpha(G)$ is defined as as the minimum of the set of ratios over all nodes

$$\alpha(G) = min(\alpha_{i_1}, \alpha_{i_2} \ldots, \alpha_{i_n})$$

Intuitively, $\alpha(G)$ indicates how close the worst performing node in G is to best response. Note that in equilibrium $\alpha(G) = 1$. We hope to obtain graphs G such that $\alpha(G)$ is as close to 1 as possible.

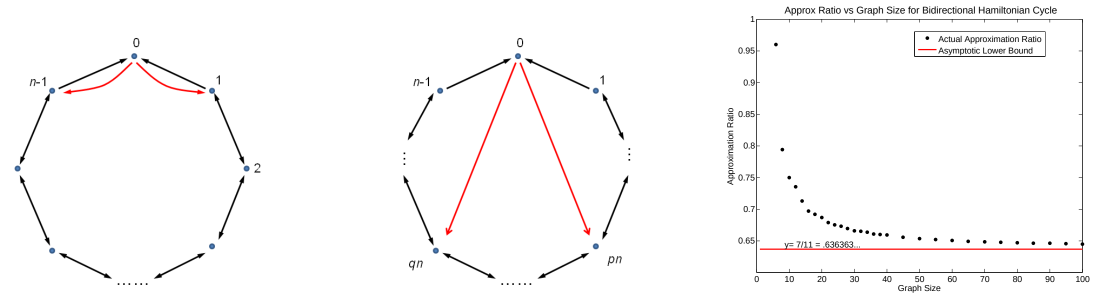
Now, for the $k = 2$ directed B^{3}C game, we demonstrate that every node in a bidirectional cycle (Figure 2(a)) will have a response value which is a $7/11$ fraction of its best response as graph size becomes arbitrarily large, thus demonstrating such an asymptotic approximation bound.

Theorem 4.1: Asymptotic bound on the approximation ratio of a bidirectional cycle $\alpha(H) = \frac{7}{11}$ as $n \to \infty$

Proof Idea: Note that H is vertex transitive, so the current response is uniform across all nodes, as is the best response. This implies $\alpha(H) = \alpha_i$ for any node $i \in H$. Now compute the value of α_i for some i, say node 0 (as shown in figure 2(a)). We'll determine node 0's current response and its best response given the fixed configuration of other players. For the current response, we carefully count all shortest paths which pass through node 0 in its current situation. We find $B_{\text{current}}(0) = \frac{1}{4}n^2 + o(n^2)$. We focus on the highest order n terms only as we are ultimately interested in the asymptotic behavior. To determine the best response, we allow node 0 to replace its current connections with edges to nodes pn and qn, where $0 < p < q < 1$, as shown in figure 2(b). We then find a closed-form function $B(p, q)$ for the new betweenness value of node 0 in terms of p and q. Using standard calculus techniques, we find the values $p*$ and $q*$ that yield the largest value for $B(p, q)$, which is the best response. Specifically, we find that $p* = 2/7$ and $q* = 5/7$, which gives a best response $B_{\text{best}}(0) = \frac{11}{28}n^2 + o(n^2)$. The final step in the proof is to compute the ratio as n gets arbitrarily large. As $n \to \infty$, we find that $\alpha_0 = \frac{1/4}{11/28}n^2 = \frac{7}{11}$, which completes the proof. A rigorous version of this sketch is documented in Appendix D.

To verify our result, we compute the approximation ratio for bidirectional cycle graphs with sizes from 10 to 100 vertices. This data is shown in figure 2(c) alongside our assymptotic bound. The data does converge toward the asymptote as n increases, which builds confidence in our approach.

Moreover, we use the approximation ratio as a mean to study how different graph structures are close to Nash equilibria. Figure 3 shows the experimental results for $k = 2$ and $k = 3$ games on several type of random graphs. The results show that purely random graphs has the worst ratio, while random Eulerian graphs in which each node has the same in-degree as the out-degree possess much better approximation ratio than other types of graphs. Further simulations for $k = 3$ comparing various quantitative properties against the approximation score yielded positive correlation between low diameter and high approximation ratio, high fairness and high approximation ratio, and

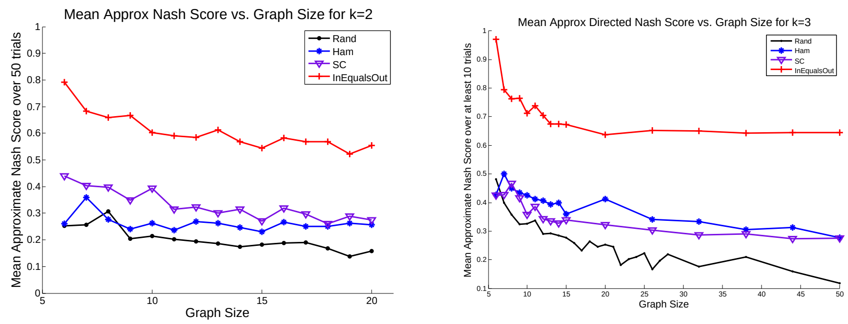


(a) Current Response

(b) Best Response

(c) $\alpha(H)$ vs. graph size

Fig. 2. Current (a) and best (b) responses for node 0 in bidirectional cycle. Plot of $\alpha(G)$ as graph size increases (c).

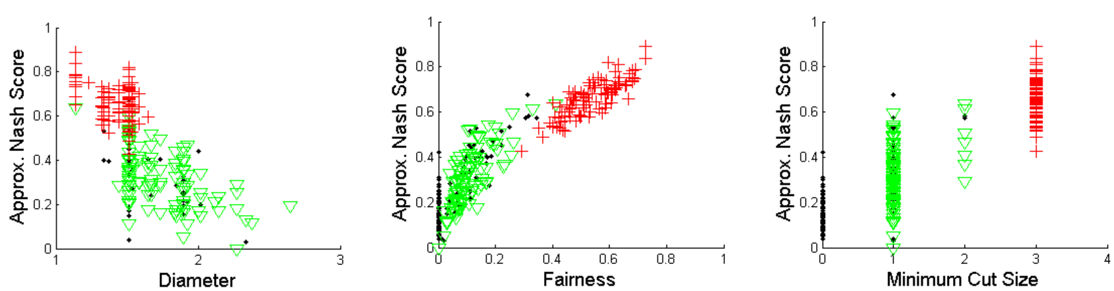


(a) $k = 2$

(b) $k = 3$

Fig. 3. Mean approximation ratio vs. graph size for various graph structures in $k = 2$ (a) and $k = 3$ (b) directed game. "Rand" means random graphs; "Ham" means graphs with Hamiltonian cycles; "SC" means strongly connected graphs; and "InEqualsOut" means Eulerian graphs with in-degree equaling out-degree.

high minimum cut size with high approximation ratio, as shown in figure 4. These experimental results suggest closer examination of vertex-transitive constant-degree expander families, such as Cayley expanders, as a direction of future study as such expanders also exhibit such properties (see [11]). However, it also remains to find a truly practical explicit construction of such families.



(a)

(b)

(c)

Fig. 4. Approximation ratio dependence on diameter (a), fairness (b), and minimum cut size (c) for $k = 3$ directed graphs with size $14 \leq n \leq 44$ arbitrarily chosen from various structural classes: Eulerian (crosses), strongly connected (triangles), and random graphs (dots). Diameter normalized by factor of $1/log(n)$.

V. Conclusion and Future Work

We have presented many analytical and experimental results regarding both pure Nash equilibria and approximate Nash equilibria for the uniform B^3C network formation game. These results provide better understanding of the B^3C game. Several conjectures on pure Nash equilibria as consistent with the results of our investigation arise: strong connectivity of all directed Nash equilibria, the existence of pure Nash equilibria with n nodes and k outgoing edges at every node, and more specifically the existence of Eulerian Nash equilibria for any n and k.

With respect to approximate Nash equilibria, all results thus far support that two types of ensembles appear to yield better approximation ratios: low uniform edge congestion (vertex-transitive expanders) and high uniform edge congestion (e.g. Abelian Cayley graphs). We wish to further explore the precise relationship between expansion and approximation to Nash equilibria given vertex-transitivity. In particular, as the approximation to Nash for the undirected game remains entirely open, exploring spectral properties of undirected graphs close to equilibrium and testing the approximation ratio of constant degree Cayley expanders (which are non-Abelian) may yield further insight.

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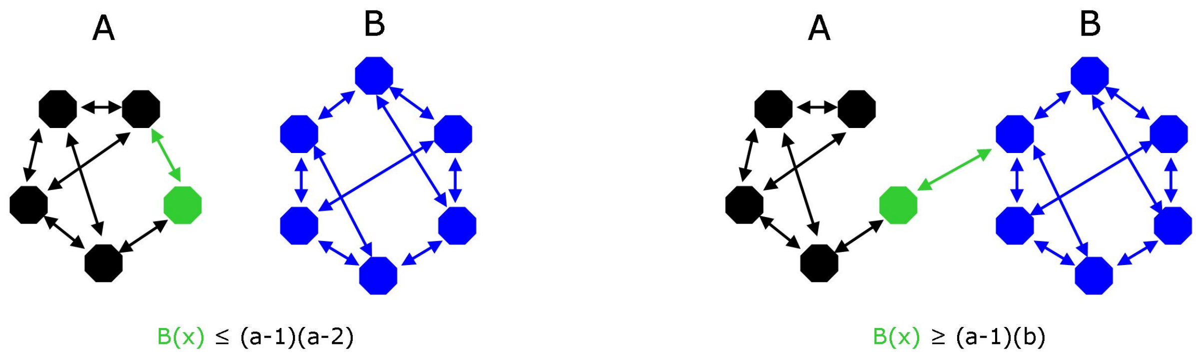


(a) Node v's current response

(b) Node v's best response

Fig. 5. Maximally connected subcomponents A and B of a non-connected undirected graph.

Appendix A
Analytical Results from Section III

In this appendix, we relate formal proofs of the theorems put forth in our analytical discussion.

A. Connectivity for Undirected Equilibria

Proof of Theorem 3.1: We prove this by contradiction. Suppose there exists an undirected graph G with vertices V and edges E which is at Nash Equilibrium but not connected. Then G has at least two maximally connected components A and B, where A and B are disjoint. This means for all $a \in A$ and for all $b \in B$, $(a, b) \notin E$. Let component A have a nodes and B have b nodes. Without loss of generality, we can suppose that $a \leq b$. Refer to the diagram in figure 5.

We first prove that there always exists an edge $e \in A$, such that if we remove e from A, A is still connected. This is because A has ak edges, k edges from each of a nodes. A must have a cycle or some multiple edges between some pair of nodes. Otherwise it is a tree which can have only $a - 1$ edges, but $ak > a - 1$ for $k \geq 1$. If A has a cycle, we can remove any edge from this cycle and A is still connected. If A has some multiple edges, we can simply remove one of them and A remains connected. So we are able to remove edge e from A without disconnecting A.

Let v be the vertex in A incident to e that built edge e. We first estimate its betweenness value $B_{\text{current}}(v)$. Notice that for any pair of nodes x, y in the graph, the shortest path from x to y can go through v only if x and y are both in A, because there is no edge connecting A and B. And there are totally $(a - 1)(a - 2)$ ordered pairs of nodes in A which do not contain v. Since each possible path can contribute at most 1 to $B(v)$, we have an upper bound $B(v) \leq (a - 1)(a - 2)$.

Now suppose v changed its strategy and removed its edge e to connect to an arbitrary node in B. Say the new edge is $e' = (v, u), u \in B$. Now every node in A can connect to every node in B, and the paths connecting them must go through this new edge e', and consequently node v. This means there will be at least $(a - 1)b$ shortest paths through v which each contribute 1 to $B(v)$. So the new betweenness value of v has a lower bound $B_{\text{new}}(v) \geq (a-1)b$. Since $a \leq b$ by definition, this implies $(a-1)b > (a-1)(a-2)$. So that $B_{\text{new}}(v) > B_{\text{current}}(v)$. Therefore, the original strategy for v is not its best response, which yields a contradicition. Thus, no undirected graph G that is not connected can be at Nash Equilibrium. ∎

(a) Node v's current response

(b) Node v's best response

Fig. 6. Maximally connected subcomponents C_1 and C_2 of a disjoint directed graph.

B. *Weak Connectivity for Directed Equilibria*

Proof of Theorem 3.2:

We prove this by contradiction. Suppose there exists a directed graph G with vertices V and edges E which is at Nash Equilibrium but is not weakly connected. First, we can cluster the nodes of G into maximal strongly connected components (SCCs), and topologically sort these components into a directed acyclic graph (DAG) of SCCs. If G is not weakly connected, as assumed, then there exist at least two such DAGs which are disjoint. We label these C_1 and C_2. Refer to the diagram in figure 6.

Next, we consider a specific type of SCC cluster: a source. We call an SCC S a *source* if for all vertices $s \in S$, $(x, s) \in E$ if and only if $x \in S$ also. We can always find a source in a DAG of SCCs, otherwise the DAG is not acyclic and therefore not a DAG. Let S_1 and S_2 be SCC sources in C_1 and C_2, respectively. We can show that at least one node in S_1 will not be at best response.

Since source S_1 is a strongly connected component, every node in S_1 has a path to every other node in S_1. Furthermore, if one or more nodes in S_1 are incident to a directed edge $e = (s, x)$ such that $s \in S_1$ and $x \in C_1$, then all nodes in S_1 have a path to x. We call x_1 the total number of such nodes in C_1 that nodes in S_1 have a path to (note that this can include other nodes in S_1). Similarly, call x_2 the total number of such nodes in C_2 that nodes in S_2 have a path to. Without loss of generality, let $x_1 \leq x_2$. We can now prove that some node v in S_1 is not at its best response.

First, we realize that v's currently has an upper bound on its betweenness centrality. To determine this bound, we must consider all possible start points and all possible end points whose path may pass through v. There are at most $s_1 - 1$ nodes where a path through v can originate, since S_1 is a source component and we cannot count v as one of these points. Furthermore, we determined that the nodes in S_1 have paths to x_1 nodes in total. If v were to lie on every single such path, there are $x_1 - 2$ possible end points, since we cannot count v or the starting point as an end point. Since each of these connections contributes at most 1 to v's betweenness, we have $B_{\text{current}}(v) \leq (s_1 - 1)(x_1 - 2)$.

Next, we discover that v can strictly increase its betweenness by removing one of its existing outgoing edges and connecting to a node in S_2 instead. If v removes one of its out-going edges, all nodes in S_1 still have a path to v, because S_1 was formerly strongly connected, and no paths from elsewhere to v were changed by the removal. So there are still $s_1 - 1$ starting points that might have a shortest path through v. Furthermore, if v connects to any node in S_2, we find that it is now the exclusive link that allows nodes in S_1 to reach all the x_2 nodes in C_2 which

i	$i+a \bmod n$	$i+b \bmod n$
1	2	4
2	3	5
3	4	6
4	5	1
5	6	2
6	1	3

TABLE I
OUT-GOING EDGE PARTNERS FOR AN ABELIAN CAYLEY GRAPH WITH $n = 6$ AND GENERATING SET $\{1, 3\}$.

nodes in S_2 can reach. So there are now at least x_2 possible end points that have a shortest path through v. This gives us a lower bound on the new response of v of $B_{\text{new}}(v) \geq (s_1 - 1)x_2$.

Recall that by we chose S_1 and S_2 such that $x_2 \geq x_1$. This implies that $B_{\text{new}}(v) > B_{\text{current}}(v)$ since $(s_1-1)x_2 > (s_1-1)(x_1-2)$. Therefore, since node v can strictly increase its betweenness from its original configuration, node v is not at a best response and the entire original graph G is not at Nash Equilibrium. This contradicts our founding assumption that a directed graph could be both stable and not weakly connected.

Therefore, all directed graphs at Nash Equilibria must be weakly connected. That is, for every pair of nodes x and y in a graph G, there must exist at least one of the edges $(x, y) \in E$ or $(y, x) \in E$. There cannot be completely disjoint components in a directed Nash Equilibrium.

■

APPENDIX B
DIRECTED PURE NASH EQUILIBRIA

A. *Exhaustive Search Results for* $n = 5, 6, 7$

We exhaustively searched and found all unique (non-isomorphic) directed Nash equilibria for $n = 5, 6, 7$. These unique stable graphs are shown in figures 7, 8, and 9.

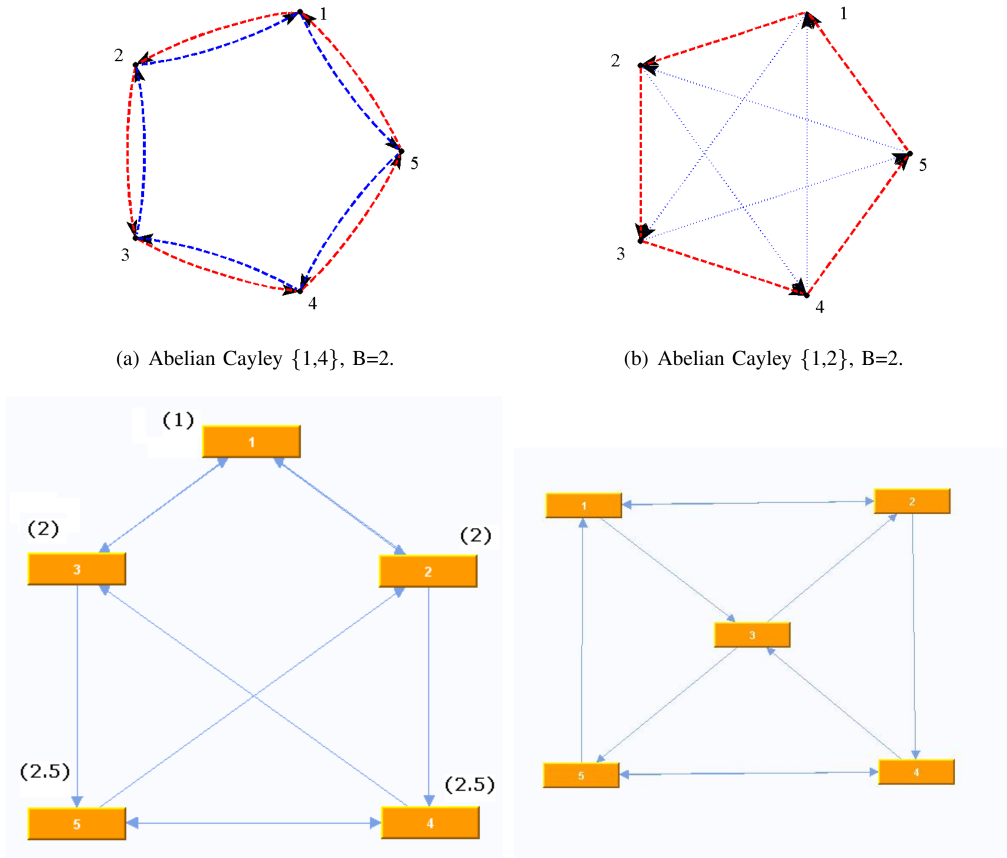
We observe may shared traits among these stable graphs, including strong connectivity, the existence of Hamiltonian cycles, and Eulerian structure (uniform in- and out- degree).

Exhaustive searches became too computationally intensive for $n \geq 8$.

B. *Construction of Directed Abelian Cayley Graphs,* $k = 2$

Here we give an example construction of a directed Abelian Cayley graph for $k = 2$. Let the Abelian Cayley graph A_n have n vertices. We choose from the group $\mathbb{Z}_n$ distinct labels for each vertex. We then assign each vertex two other vertices to build out-going edges to since $k = 2$. This assignment is performed using addition modulo n with a two member generating set $\{a, b\}$, as we define addition as the group operation for all constructions. Thus, the node labeled i connects out-going edges to $i + a$ and $i + b$.

A full construction for the graph A_6 with generating set $\{1, 3\}$ is given in table I. Note that it does not matter that we have chosen to label the nodes 1 through 6, rather than 0 through 5. The definition of Abelian Cayley still applies. The diagram of this graph is shown in 8(a).



(a) Abelian Cayley {1,4}, B=2.

(b) Abelian Cayley {1,2}, B=2.

(c) Asymmetric, B=1,2,2,2.5,2.5.

(d) Asymmetric, B=2,2,2,2,2.

Fig. 7. All unique Nash equilibria for $n = 5$. Generating sets and betweenness are indicated.

C. Vertex Transitivity Proof for Cayley Graphs

Theorem B.1: Let G be a group and $S \subset G - \{1\}$. Let the Cayley graph be $\Gamma = \Gamma(G, S)$, such that $V(\Gamma) = G$ and $E(\Gamma) = \{(g, gs) : g \in G, s \in S\}$. All Cayley graphs are vertex transitive.

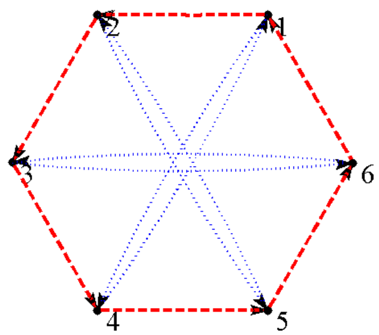
Proof: First we show that there exists a left translation of the Cayley graph $i_a(u) = au$, which is an automorphism for all $a \in S$ and $u \in G$. For any $(u, v) \in E(\Gamma)$, we have $u^{-1}v \in S$. Thus $i_a(u)^{-1}i_a(v) = u^{-1}a^{-1}av = u^{-1}(a^{-1}a)v = u^{-1}v \in S$, and $[i_a(u), i_a(v)] \in E(\Gamma)$. So $i_a(u)$ is an automorphism. Therefore, $i_a(u)$ can map any pair of vertices in G. For example, for some pair of vertices m and n, let $a = nm^{-1}$. Then $i_a(m) = am = nm^{-1}m = n$. It takes m to n. Therefore, all Cayley graphs are vertex transitive. ■

D. Isomorphism Results for Directed Abelian Cayley Graphs, $k = 2$

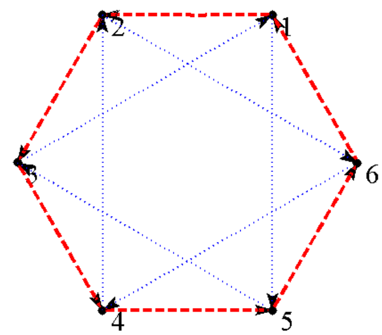
Theorem B.2: Let G be an Abelian Cayley graph of size n with generating set $\{a, b\}$. If $gcd(a, n) = 1$, then we may relabel the vertices of G to an isomorphic graph G' that is rotationally symmetric for parameters $\{1, c\}$, where c is some constant $1 < c < n$.

Proof:

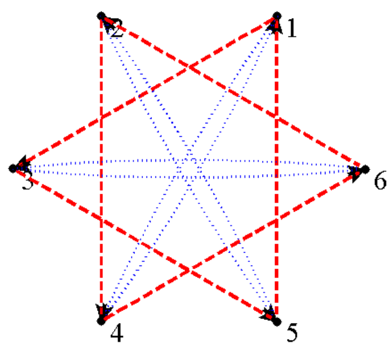
Let G have original vertices labeled $0, 1, 2, 3, ...n - 1$. Because G is rotationally symmetric for some a, b, we



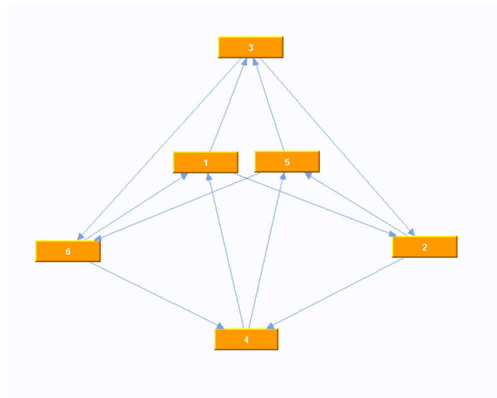
(a) Abelian Cayley {1,3},B=4,4,4,4,4,4.



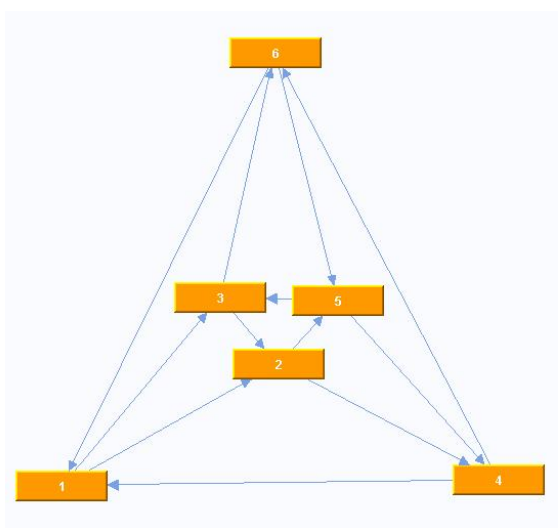
(b) Abelian Cayley {1,4},B=4,4,4,4,4,4.



(c) Abelian Cayley {2,3},B=4,4,4,4,4,4.

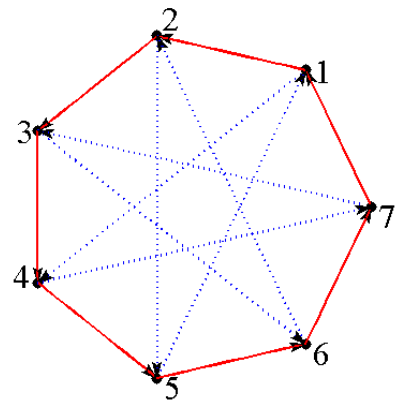


(d) Asymmetric,B=3.5,3.5,2,2,3.5,3.5.



(e) Asymmetric, B=3.5,3.5,2.5,3.5,2.5,2.5.

Fig. 8. All unique Nash equilibria for $n = 6$. Generating sets and betweenness are indicated.



(a) Abelian Cayley {1,3}, B=5,5,5,5,5,5,5.

Fig. 9. All unique Nash equilibrium for $n = 7$. The generating set and betweenness are indicated.

know for each node i that the edges $(i, i + a)$ and $(i, i + b)$ exist. Also, each node of G can be written in the form aj mod n, where $0 \leq j < n$.

To relabel G as G', we define a relabeling function f as follows

$$f(aj \bmod n) = j \quad \text{for } 0 \leq j \leq n - 1$$

This means for each j, we relabel the vertex $aj \mod n$ as vertex j. Remember, a is a constant relatively prime to n.

Tracing out this transformation for a few values of j, we see that

$$\begin{aligned} f(aj \bmod n) &= j \\ f(0) &= 0 \\ f(a) &= 1 \\ f(2a) &= 2 \\ f(3a) &= 3 \end{aligned}$$

From this trace we easily see that each vertex pair $(i, i + a)$ will be relabeled as some unique $(j, j + 1)$ where $i = aj \bmod n$. Thus, we have transformed one of the rotationally symmetric parameters from a to 1. Note that this unique relabeling is only true if a and n are relatively prime. If a and n are not relatively prime, the relabeling is non-unique and the relabeling function is invalid.

Now, in order to maintain the rotational symmetry of the relabeled graph G', we find the other constant parameter c such that for each vertex i' in G' there exists an edge $(i', i' + c)$. Since this edge would in the original graph G be isomorphic to $(i, i + b)$, we find that the following condition must hold

$$(f(i) + c) \bmod n = f((i + b) \bmod n)$$

According to our relabeling scheme, $i = aj \bmod n$ for some j and $f(aj \bmod n) = j$. We may thus rewrite our equation as

$$\begin{aligned}(f(i) + c) \bmod n &= f((i + b) \bmod n)\\ (f(aj \bmod n) + c) \bmod n &= f(((aj \bmod n) + b) \bmod n)\\ (j + c) \bmod n &= f(((aj \bmod n) + b) \bmod n)\end{aligned}$$

We realize that we may rewrite b as $b = aj' \bmod n$ for some constant integer j' such that $0 \leq j' \leq n - 1$. So we have

$$\begin{aligned}(j + c) \bmod n &= f(((aj \bmod n) + b) \bmod n)\\ (j + c) \bmod n &= f(a(j + j') \bmod n)\end{aligned}$$

Executing our relabeling function, we see that $f((a(j + j')) \bmod n)$ = $(j + j') \bmod n$. So we have

$$(j + c) \bmod n = (j + j') \bmod n$$

Thus we find that $c = j'$. This means there exists a constant c which will make G' rotationally symmetric for (1,c).

To find c, we know that $b = aj' \bmod n$, which implies $b = ac \bmod n$. Solving for c gives

$$c = \frac{b + xn}{a}$$

where x is the smallest possible positive integer which makes the expression $\frac{b+xn}{a}$ an integer value. ■

E. *Construction of Partitioned Abelian Cayley Nash Equilibria*

To see how these graphs are constructed, consider Figure 10. For this graph we have two generating sets $A = \{1, 2\}$ and $B = \{2, 7\}$. We use set A to determine the out-going edges for nodes labeled $1, 3, 5, \ldots$ and set B to find the out-going edges for nodes $2, 4, 6, \ldots$. Notice how this method $partitions$ the vertices, which gives the name *Partitioned Abelian Cayley*.

Partitioned Abelian Cayley graphs can be generalized for higher numbers of generating sets, as we shall see below. A stable partitioned Abelian Cayley graph built with three generating sets is shown in figure 11. The only restriction is that the number of generating sets must divide the order of the graph.

The properties of stable, directed, partitioned Abelian Cayley graphs are interesting. All we have discovered so far are strongly connected. However, the other properties common to Abelian Cayley stable graphs are not always shared. In particular, partitioned Abelian Cayley graphs can have low fairness and still be stable. For example, the stable graph in figure 11 has fairness of $1/3$.

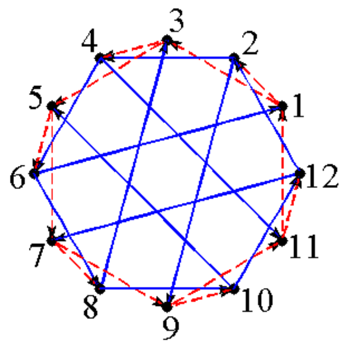
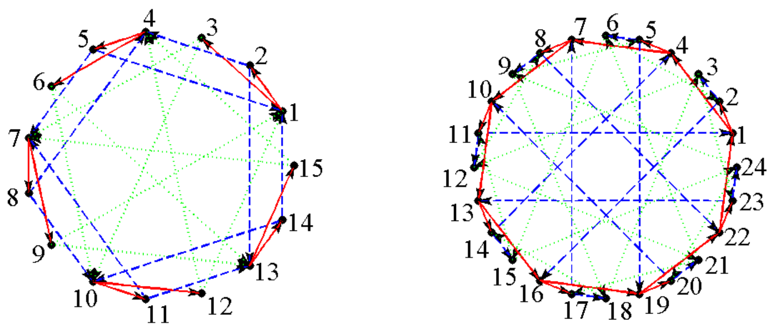


Fig. 10. A Partitioned Abelian Cayley Graph, n=12. Generating sets: $\{1, 2\}$; $\{2, 7\}$.



(a) Partitioned Abelian Cayley, n=15. Generating sets: $\{1, 2\}$, $\{2, 11\}$, $\{4, 7\}$

(b) Partitioned Abelian Cayley, n=15. Generating sets: $\{1, 3\}$, $\{1, 14\}$, $\{9, 20\}$

Fig. 11. Partitioned Abelian Cayley Graphs with 3 generating sets.

F. Isomorphism Results for Paritioned Abelian Cayley Graphs

In order to lower the search space, we have explored cases where partitioned Abelian Cayley are isomorphic to each other. Below are results we have established for graphs with two generating sets. Further, and perhaps more far-reaching, results are the subject of future research.

Theorem B.3: Consider a 2-generator-set Partitioned Abelian Cayley graph (denoted by G) of size $n = 2m$ for some $m \in \mathbb{Z}$. The generator sets are of the following form:

Generating set for odd nodes: $\{a, b\}$	a odd, b even
Generating set for even nodes: $\{a', b'\}$	a' odd, b' even

Then G is isomorphic to another 2-generator-set Abelian Cayley graph, G' with sets:

Generating set for odd nodes: $\{a, b'\}$	a odd, b' even
Generating set for even nodes: $\{a', b\}$	a' odd, b even

In other words, G is isomorphic to a graph where the even numbers in the generating sets have been switched.

Proof: Define a mapping $\phi : G \to G'$ by:

$$\phi(i) = \begin{cases} i + a & \text{if } i \text{ is odd} \\ i + a' & \text{if } i \text{ is even} \end{cases}$$

Note that throughout this proof, all addition operations are modulo n. We first show that ϕ is one-to-one and onto. Note that if $i + a = j + a$, then by cancelation, $i = j$ (mod n of course). Thus ϕ is one-to-one. To show ϕ is onto, consider an arbitrary node $v \in V(G')$. If v is odd, then the equation $v - a' \bmod n$ will give you its pre-image. If v is even, then $v - a \bmod n$ gives you the pre-image. Thus ϕ is onto.

Now we must verify that ϕ preserves the structure of the graph. That is, we must show that for u and v in $V(G)$, if (u, v) is an edge of G, then $(\phi(u), \phi(v))$ is an edge of G'.

Let's first consider the case where i is even. In G, The neighbors of i are $i + a'$ and $i + b'$. $\phi(i) = i + a'$ is a vertex of G'. Since i is even and a' is odd, it follows that $i + a'$ is odd, thus its neighbors in G' are $(i + a') + a$ and $(i + a') + b'$.

Now we look at the image of i's neighbors. $i + a'$ is odd, so $\phi(i + a') = (i + a') + a$. $i + b'$ is even, so $\phi(i + b') = (i + b') + a'$. Thus the graph structure is preserved when i is even.

Now consider the case where i is odd. His neighbors in G are $i + a$ and $i + b$. $\phi(i) = i + a$. $i + a$ is even, so in G' the neighbors of $i + a$ are $(i + a) + a'$ and $(i + a) + b$.

What are the images of i's neighbors? $i + a$ is even, so $\phi(i + a) = (i + a) + a'$. $i + b$ is odd so $\phi(i + b) = (i + b) + a$. So the graph structure is preserved when i is odd also.

This shows that G is isomorphic to G'. Note that the result also holds in the following case:

Generating set for odd nodes: [a,b]	a even, b odd
Generating set for even nodes: [a',b']	a' even, b' odd

Switching the even numbers in the generating set produces a graph isomorphic to the one with the above generating sets. Just define the mapping as follows:

$$\phi(i) = \begin{cases} i + b & \text{if } i \text{ is odd} \\ i + b' & \text{if } i \text{ is even} \end{cases}$$

■

Theorem B.4: Let G be a 2-generating-set Partitioned Abelian Cayley graph. Let the generating sets be denoted by $\{a, b\}$ and $\{a', b'\}$. Then, G is isomorphic to a graph, G', with generating sets $\{n - a', n - b'\}$ and $\{n - a, n - b\}$.

Proof:

Define the mapping $\phi : V(G) \to V(G')$ by $\phi(i) = n - i + 1$. We first show ϕ is one-to-one and onto. If $\phi(i) = \phi(j)$, then $n - i + 1 = n - j + 1$. By cancelation, $i = j$. So ϕ is one-to-one. To show ϕ is onto, note that the vertices of G' are the numbers 1 through n. Since i is always less than or equal to n, and i is also any number between 1 and n, it is clear that for any $j \in V(G')$, we can determine an i that maps to it.

For i in G, first assume that i is odd (also recall that n is even). Then $\phi(i) = n - i + 1$ is even. Thus in G', $n - i + 1$ maps to the nodes $n - i + 1 + n - a$ and $n - i + 1 + n - b$.

In G, i maps to $i + a$ and $i + b$. Looking at the images of these nodes, we see that $\phi(i + a) = n - i - a + 1$ and $\phi(i + b) = n - i - b + 1$. Then, we see that

$$n - i + 1 + n - a = 2n - i - a + 1 = n - i - a + 1 = \phi(i + a)$$

because all operations are modulo n. Similarly,

$$n - i + 1 + n - b = 2n - i - b + 1 = n - i - b + 1 = \phi(i + b)$$

For i even, the result is similar. ∎

Appendix C
Undirected Pure Nash Equilibria

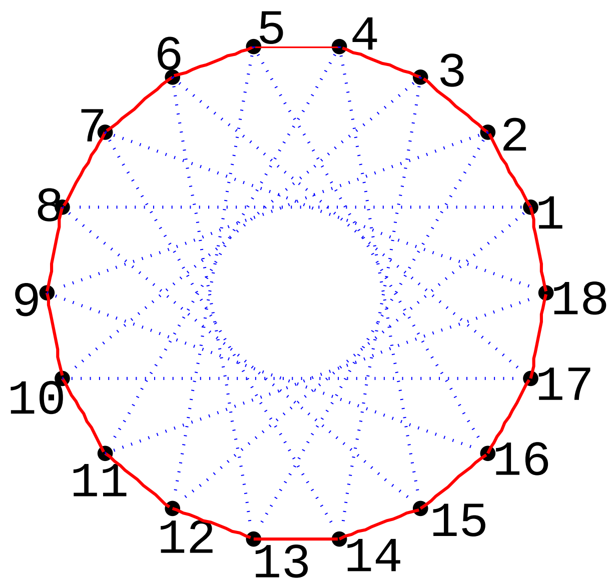


Fig. 12. A Rotationally Symmetric Undirected Nash Graph for $n = 18$.

Appendix D
An Asymptotic Bound on Approximation Ratio of a Bidirectional Cycle

In this section, we show that any bidirectional Hamiltonian cycle graph (an Abelian Cayley graph with generating set $\{1, n-1\}$) has such an asymptotic bound. We find that the approximation ratio $\alpha = 7/11$ as n goes to infinity.

As a first step in this proof, we derive a closed form expression for the current response of any node in such a graph. Next, we provide an asymptotic value for the best response of any node in such a graph. Finally, we show that the approximation ratio will be an asymptotic constant.

A. *Closed Form Expression for Current Response*

Because a bidirectional hamiltonian cycle is vertex transitive, we may compute the current response of any node by looking at just a single node. We may thus consider some node i in the graph of size n. To determine node i's betweenness, we must calculate the number of shortest paths which go through i. We can break these down by length, and count them as follows

Lemma D.1: The number of shortest paths of length l which pass through i, denoted $NSP_i(l)$, in a bidirectional hamiltonian cycle, is given by

$$NSP_i(l) = l - 1$$

Proof Idea Given a fixed l, we see that any shortest path this length must include l edges and $l+1$ nodes. Node i cannot be a start or end point on this path if the path is to contribute to its betweenness, so it may be any one of the $l-1$ middle points in the path. Therefore, $NSP_i(l) = l - 1$.

We know unique shortest paths through i will have length $1 < l < d$, where d is the diameter of the graph. We may now calculate the total number of unique shortest paths through i with less than diameter length as $\sum_{l=2}^{d-1} l - 1$.

We next consider paths which have diameter length, $l = d$. This must be considered differently based on whether the graph size n is even or odd. If n is odd, these paths will still be unique, since for nodes s, t which are one diameter apart there is only one such path of length $l = d$. If n is even, these paths are non-unique, because there exist 2 paths of length $l = d$ between s and t and only 1 goes through i. So we have

If n is odd:

$$NSP_{i,\text{odd, unique}} = \sum_{l=2}^{d} (l - 1)$$

If n is even:

$$NSP_{i,\text{even, unique}} = \sum_{l=2}^{d-1} (l - 1)$$

$$NSP_{i,\text{even, non-unique}} = d - 1$$

We know that each unique path from s to t contributes $+2$ to the betweenness value of i, $+1$ for $s \to t$ and $+1$ for $t \to s$. We also know that each non-unique path contributes $+1$ to the betweenness of i, $+1/2$ for $s \to t$ and $+1$ for $t \to s$. We thus have

$$B(i) = 2 \cdot NSP_{i,\text{unique}} + NSP_{i,\text{non-unique}}$$

So if n is odd:

$$B_{\text{odd } n}(i) = 2 \cdot \sum_{l=2}^{d} (l - 1)$$

If n is even:

$$B_{\text{even } n}(i) = 2 \cdot \sum_{l=2}^{d-1} (l - 1) + d - 1$$

Computing these sums gives

$$B_{\text{odd } n}(i) = \quad 2(1 + 2 + \cdots + d - 2 + d - 1) = \quad d(d - 1)$$

$$B_{\text{even } n}(i) = \quad 2(1 + 2 + \cdots + d - 2) + d - 1 = \quad (d - 1)^2$$

Thus, we have the current response

$$B_{\text{current}}(i) = \begin{cases} d(d-1) & \text{if } n \text{ is odd} \\ (d-1)^2 & \text{if } n \text{ is even} \end{cases}$$

Since $d = n/2$ when n is even and $d = (n-1)/2$ when n is odd, we may say

$$B_{\text{current}}(i) = \begin{cases} \frac{n-1}{2}(\frac{n-1}{2} - 1) & \text{if } n \text{ is odd} \\ (\frac{n}{2} - 1)^2 & \text{if } n \text{ is even} \end{cases}$$

Expanding gives

$$B_{\text{current}}(i) = \begin{cases} \frac{1}{4}n^2 - n + \frac{3}{4} & \text{if } n \text{ is odd} \\ \frac{1}{4}n^2 - n + 1 & \text{if } n \text{ is even} \end{cases}$$

B. Asymptotic Expression for Best Response

Since the graph is vertex transitive, we only need to analyze one node, namely vertex 0. We start with some preliminary results before proving the overall result. For these lemmata, we suppose vertex 0 has only one outgoing edge, namely $(0, qn)$, where $0 < q < 1$. We may use this simpler case to establish basic results. Later, we consider allowing node 0 two outgoing edges, $(0, qn)$ and $(0, pn)$, where $(0 < p, q < 1)$. Finally, we can show an asymptotic bound for the approximation ratio of the graph G.

1) ***Node 0 has 1 outgoing edge,*** $(0, qn)$ *:* For the $k = 1$ scenerio, we must determine how many paths starting at vertex s and ending at vertex t will flow through 0. We use $s = an$ and $t = bn$, where $0 < a, b < 1$ as parameters to assist our calculations. We then exhaustively consider all possible configurations for q, a, b to find the betweenness centrality of node 0 in this case.

Case 1 $0 < a < b < q < 1$

Lemma D.2: If $0 < a < b < q$ the betweenness gained for vertex 0 is

$$\frac{q^2}{8}n^2 + o(n^2)$$

Proof: In this case, one path would be $an \to an + 1 \to \ldots \to bn$. The other path is $an \to an - 1 \to \ldots \to 0 \to qn \to qn - 1 \to \ldots \to bn$. The length of these two paths are $(b - a)n$ and $an + 1 + (q - b)n$. In order to let latter one be the shorter one, we need $b - a > a + q - b$, which implies $b - a > q/2$.

In order to calculate the number of pairs that satisfy this condition, we first fix $bn - an = l$. Then there are totally $qn - l$ valid pairs of vertices. Because an can be any vertex within $(0, qn - l)$, $bn = an + l$ is then fixed. And we have $qn/2 < l < qn$. So the total number would be $\sum_{l=qn/2}^{qn} qn - l = \frac{q^2}{8}n^2 + o(n^2)$.

■

Case 2 $0 < a < q < b < 1$

Lemma D.3: If $0 < a < q$ and $q < b < 1$, then the betweenness gained for vertex 0 is

$$\frac{q(1-q)}{2}n^2 + o(n^2)$$

Proof: In this case, there are two paths from s to t. One is $s \to s+1 \to \ldots \to t$, whose length is $(b-a)n$. The other one is $s \to s-1 \ldots \to 0 \to qn \to qn+1 \to \ldots \to t$, whose length is $an+1+(b-q)n$. In order to let the latter one which will pass through vertex 0 be the shorter one, we need to have $b-a > a+b-q$, which is $a < q/2$. Also we have $q < b < 1$. So the betweenness value contributes to vertex 0 here is $(q/2)n\cdot(1-q)n+o(n^2) = \frac{q(1-q)}{2}n^2+o(n^2)$. ■

*2) **Node 0 has 2 outgoing edges,** $(0, qn)$ **and** $(0, pn)$:* We analyze the $k = 2$ scenario through exhaustive consideration of all possible configurations of p, q and a, b. Again, we denote the starting point s and terminal point t as $s = an$ and $t = bn$, where $0 < a, b < 1$.

Case 1 $0 < a, b < q$ and its symmetric opposite, $p < a, b < 1$

This case itself divides into separate cases:

1) $0 < b < a < q$ In this case, s will always connect directly to t, and never go through 0. So the betweenness contributed here is 0.
2) $0 < a < b < q$ Here, if the path $s \to t$ goes through 0, it will only use the q edge. This is the case described in Lemma D.2 above. So the betweenness gained is

$$\frac{q^2}{8}n^2 + o(n^2) \tag{2}$$

3) Similarly, if $p < a < b < 1$ then the betweenness contribution is 0. Also, if $p < b < a < 1$, we relabel the nodes in the counter-clockwise direction, hence p becomes $1 - p$. Now this case is identical to part 2 above, so we have

$$\frac{(1-p)^2}{8}n^2 + o(n^2) \tag{3}$$

Adding 2 and 3, we get the total betweenness gained for Case 1:

$$B_1(0) = \frac{q^2}{8}n^2 + \frac{(1-p)^2}{8}n^2 + o(n^2) \tag{4}$$

Case 2 $0 < a < q < b < p$

Here, we consider all possible paths s might use to get to t. We note that the length of the direct path from s to t is $(b - a)n$. Also, the length of the path $s \to qn \to t$ is $an + 1 + (b - q)n$. Finally, the length of the path $s \to pn \to t$ is $an + 1 + (p - b)n$.

Now, if the shortest path from s to t goes through 0, then one of the following must be true:

$$a + b - q < b - a \quad \Longrightarrow \quad a < q/2$$

or

$$a + p - b < b - a \quad \Longrightarrow \quad a > q/2 \quad \text{and} \quad p/2 < b - a$$

If $a < \frac{q}{2}$, then the betweenness gained is

$$\frac{q}{2}(p - q)n^2 \tag{5}$$

This is because there are $\frac{q}{2}n$ choices for s and $(p - q)n$ choices for t.

If $a > \frac{q}{2}$, it is obvious that the q edge will never be used, so we ignore it. Thus we can combine this case with the case where a satisfies $q < a < p$. So for the remaining part of this discussion for **Case 2**, we can suppose

$0 < a < p$.

To analyze this situation, we fix a length $l = (b-a)n$, and note that an must satisfy $\frac{q}{2}n < an < (p-l)n$. This is because if $an > (p-l)n$, this would imply that $bn > pn$, which is not allowed for this case. Thus there are $(p - \frac{q}{2} - l)n$ possible start points ('s' nodes) for this case. Also, for each fixed l, there is exactly one corresponding end point (since $l = (b-a)n$). So the expression $(p - \frac{q}{2} - l)n$ represents one path for a certain l. Summing over all possible l in this case, we get:

$$\sum_{l=\frac{pn}{2}}^{pn-\frac{qn}{2}} (pn - qn/2 - l) = \sum_{x=0}^{(\frac{p-q}{2})n} x = \frac{(p-q)^2}{8} n^2 \tag{6}$$

Where $x = pn - \frac{qn}{2} - l$.

Similarly, this argument also holds for $q < b < p < a$. In this case, replace q by $1-p$ and p by $1-q$ in equations 5 and 6 above. Therefore, the total betweenness gained for **Case 2** is

$$\begin{aligned} B_2(0) &= \frac{q}{2}(p-q)n^2 + \frac{(p-q)^2}{8}n^2 + \frac{(p-q)^2}{8}n^2 + \frac{(1-p)}{2}(p-q)n^2 \\ &= (p-q)\left[\frac{1}{2} - \frac{(p-q)}{4}\right] n^2 + o(n^2) \end{aligned} \tag{7}$$

Case 3 $\quad 0 < a < p$, $p < b < 1$

Notice here that we are combining $0 < a < q$ and $q < a < p$. This is because for both of these cases, $p < b < 1$, thus any shortest path from an to bn that passes through 0 would never use the q edge. Only the p edge will be used to get to bn. So we may disregard the q edge. With that, we may now use Lemma D.3. Thus, the betweenness gained is

$$\frac{p(1-p)}{2}n^2 + o(n^2) \tag{8}$$

Similarly, if $p < a < 1$ and $0 < b < q$, the p edge will never be used, so we have the following betweenness contribution

$$\frac{q(1-q)}{2}n^2 + o(n^2) \tag{9}$$

Thus the total betweenness for **Case 3** is

$$B_3(0) = \frac{p(1-p)}{2}n^2 + \frac{q(1-q)}{2}n^2 + o(n^2) \tag{10}$$

Adding up all the betweenness contributions, namely those of equations 4, 7, and 10, we have the following expression for the betweenness of the best response

$$B(0) = \left[\frac{q^2}{8} + \frac{(1-p)^2}{8} + (p-q)\left(\frac{1}{2} - \frac{(p-q)}{4}\right) + \frac{p(1-p)}{2} + \frac{q(1-q)}{2}\right] n^2 + o(n^2) \tag{11}$$

The coefficient of n^2 in the above equation can be viewed as a function of p and q. We wish to maximize this value in order to obtain the best response. Taking partial derivatives we have

$$\frac{\partial f}{\partial p} = \frac{3}{4} - \frac{5p}{4} + \frac{q}{2} = \frac{p}{2} - \frac{5q}{4}$$

Setting the partials equal to zero, we find we achieve a maximum when $p = 5/7$ and $q = 2/7$. Plugging these values into 11, we find that

$$B_{\text{best}}(0) = \frac{11}{28}n^2 + o(n^2)$$

C. Computing Approximation Ratio

Recall that the current betweenness of node 0 in a bidirectional Hamiltonian cycle is

$$\begin{aligned} B_{\text{current}}(0) &= \begin{cases} \frac{1}{4}n^2 - n + \frac{3}{4} & \text{if } n \text{ is odd} \\ \frac{1}{4}n^2 - n + 1 & \text{if } n \text{ is even} \end{cases} \\ &= \frac{1}{4}n^2 + o(n^2) \end{aligned}$$

Thus we have the approximation ratio asymptotically bounded by

$$\alpha = \frac{B_{\text{current}}}{B_{\text{best}}} = \frac{\frac{1}{4}n^2 + o(n^2)}{\frac{11}{28}n^2 + o(n^2)} = \frac{7}{11} \quad \text{as } n \to \infty$$

QED.