Numerical Analysis of Focal Length of a Simple Electromagnetic Lens

Marc Sweetgall and Michael C. Hughes, Class of 2010

Olin College of Engineering

***Abstract*— Often used in electron microscopy, an electromagnetic lens utilizes the magnetic field generated by a coil of current-carrying wire to focus a beam of electrons onto a desired point. In this work, we propose a simple model of an electromagnetic lens, apply the optical concept of focal length to the electromagnetic domain, and propose two distinct methods for calculating focal length using a numerical simulation of the lens. We then use the results of systematic experimentation with the simulation to determine how the focal length of a given model lens depends mathematically upon the lens' geometry, the applied current, and the velocity of the electron beam.**

I. Introduction

The advent of electron microscopes in the last century provided scientists with a uniquely powerful tool for examining matter at minute scales invisible to the naked eye. Applications of this technology range from detailed examination of protein molecules in biology to the study of an alloy's crystalline structure in materials science.

Electron microscopes work by shooting a controlled cone-like beam of electrons from an electron gun toward a specimen. Along the way, this beam of electrons encounters the magnetic field produced by an electromagnetic lens, which exerts a focusing force upon the beam electrons so that they strike the specimen at a desired point. After the beam reaches the specimen, a variety of techniques can be used to generate a usable image, such as transmission and scanning.

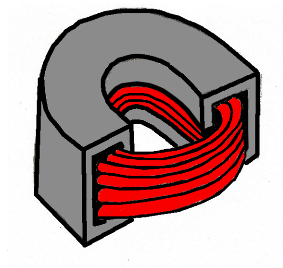
In this work, we examine the focusing process of the electromagnetic lens. Specifically, we calculate the focal length of the lens under a variety of lens and beam conditions and determine the mathematical relationships that govern how the focal length depends on the lens' geometry, the applied current, and the electron beam's velocity.

First, we briefly discuss the properties of a real-world electromagnetic lens and propose a simplified model for use in our analysis. Next, we introduce the concept of focal length as used in optical converging lenses, look at two different ways to calculate focal length, and apply these concepts to the electromagnetic lens. Finally, we utilize a numerical simulation of a simplified electromagnetic lens to determine mathematically how focal length depends on various parameters of the lens and incident electrons.

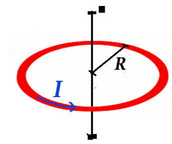
II. Model

A. The Lens

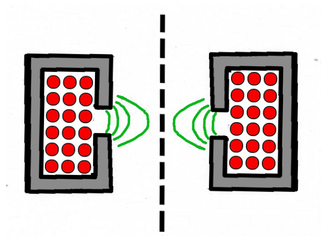
As used in electron microscopy, an electromagnetic lens consists of loops of electrical wire wound in a circular coil. When a current is applied to the coil, the moving electrons within the wire will generate a



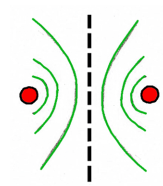
(a) Real Lens 3D



(b) Model Lens 3D



(c) Real Lens Cross-Section



(d) Model Lens Cross-Section

Fig. 1. Visual comparison between real and model electromagnetic lenses. Three dimensional diagrams are shown of both the realistic electromagnetic lens (a) and the simplified model of the lens (b). In the realistic version, we note that the current carrying wire (in red) is enclosed in an iron shield (in grey), which has been partially cut-away to more easily see the wire inside. The model simplifies reality by considering the lens as just one infinitely thin loop of wire with radius R, centered on the z-axis and carrying current I. Cross-sectional views revealing each lens' magnetic field lines (in green) are also visible for both the realistic lens (c) and model (d). We note that the realistic lens' shield contains the field generated by the coil within the desired air gap. In contrast, our model's unshielded field extends over an infinite range.

magnetic field around the coil. This magnetic field will exert a force on electrons passing through the coil, causing their paths to cross at one point along the center axis of the coil. Thus, the coil acts as an electromagnetic lens since it forces electrons toward a single convergence point.

In real-world applications, the coil of the lens is almost always surrounded by an iron shield, as shown in Figure 1(a). The iron shield contains the magnetic field produced by the coil, preventing it from escaping except over a very small region within a gap in the center of the shield. This resulting field can be seen in Figure 1(c). The shield's gap is machined with a high degree of accuracy so that the field exists only in the desired region of the lens, which substantially reduces edge effects. The full purpose of this shield will be discussed later.

In developing a numerical model for the electromagnetic lens, we chose to neglect the effects of the iron shield because we were unable to gain any traction on a mathematical method for calculating the resulting magnetic field. Instead, we decided to simplify the lens as a single circular loop of wire with radius R carrying current of I amps. Our simplified model can be visualized in Figures 1(b) and 1(d).

To calculate the total magnetic field generated by the simplified lens at an arbitrary point in space $\vec{r}$, we calculate the field contribution of each infinitely small piece using Biot-Savart's Law and then sum up all the contributions to find the total field via superposition. A coordinate free expression of this calculation is found in Equation (1),

$$\vec{B}(\vec{r}) = \int_C d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_C \frac{\vec{I}\times(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}ds \quad (1)$$

where $\vec{I}$ denotes wire's current vector and $\vec{r'}$ gives the absolute position of the minute point on the coil whose field contribution is being calculated.

In order to make Equation (1) computationally tractable, we establish our coil's center as the z-axis and define a suitable parametrization of $\vec{I}$ and $\vec{r'}$. We can express these variables using R, the loop's radius and θ, the angular location of a piece of a loop. The resulting expression thus becomes tractable using numerical computation methods.

B. The Electron Trajectory

Now that we can calculate the total magnetic field due to the coil at any position in space, we can compute the path of an electron as it travels through the loop. The force exerted upon an electron with velocity $\vec{v}$ at a given point $\vec{r}$ in the magnetic field $\vec{B}$ is given by the Lorentz Force Law, given in Equation (2).

$$\vec{F} = q(\vec{v}\times\vec{B}(\vec{r})) \quad (2)$$

Combining Equation (2) with Newton's Second Law of Motion, we can calculate an electron's instantaneous velocity and position throughout its journey through the lens using a numerical differential equation solver.

An important result of Equation (2) is the fact that the cross product in the Lorenz Force Law results in a force perpendicular to the direction of motion. This means that the field can never do any work, and the magnitude of the velocity of the electron should remain constant through its trajectory.

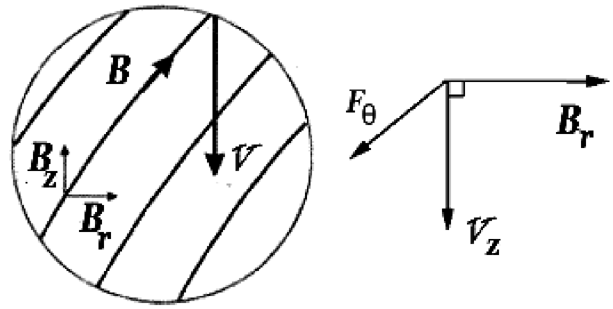


Fig. 2. Diagram showing the initial interaction between an electron with velocity $\vec{v}$ and lens' magnetic field B. Note the decomposition of $\vec{B}$ into only $\hat{r}$ and $\hat{z}$ components, since its $\hat{\theta}$ component is always zero. The resulting force of this initial interaction acts in the $\hat{\theta}$ direction, pushing the electron into a helical trajectory as it travels through the lens.

C. Focusing

As mentioned earlier, electrons fired through the electromagnetic lens with the appropriate initial conditions will converge at a single point on the other side of the lens. Generally, the appropriate conditions are those which cause the electron to enter the lens at an almost perpendicular angle to the plane of the coil. Ideally, the electrons also hit the lens coil near its center where the magnetic field lines follow the optical axis, in this case the z-axis. Under these conditions, focusing results due to the influence of the Lorentz Force Law, Equation (2). We can use this law to trace out qualitatively the path of an electron which enters the lens' field off the optical axis and is gradually forced to cross that axis. For the purposes of this trajectory walk-through, we will decompose all vectors into cylindrical coordinate components.

We can visualize a close look at the electron's initial reaction to the field with Figure 2. Our first observation is that the magnetic field of the entire lens has $\vec{B_r}$ and $\vec{B_z}$ components, but no $\vec{B_\theta}$ component. Next, we see that upon entry to the magnetic field slightly offset from the optical axis, an electron with velocity mostly in the $\hat{z}$ direction will, via the Lorentz Force Law, feel a force proportional to qv_zB_r in the $\hat{\theta}$ direction. This $\hat{\theta}$ component of acceleration leads to the helical motion characteristic of magnetic fields. Furthermore, as soon as the electron gains a $\vec{v_\theta}$ component, it additionally feels a force along the radius direction $\hat{r}$ equal to $-qv_\theta B_z$, which thus causes the electron to accelerate toward the optical axis. As long as the electron continues in helical motion and remains in a region of appreciable field strength, both the $\hat{\theta}$ component of velocity and $\hat{z}$ component of $\vec{B}$ will exist, pushing the electron toward the z axis and thus focusing it. For additional discussion of this behavior, please refer to Goodhew, et. al. [1].

D. Focal Length

The behavior of the electrons in the electromagnetic lens is analogous to the behavior of light in a convergent optical lens. This allows us to apply the concept of focal length to the electromagnetic domain as well as borrow optical methods for computing this length for our lens. Qualitatively, focal length is a measure of a lens' power, since lenses with shorter focal lengths bend the incoming beam more strongly toward the optical axis, while lenses with longer focal lengths do not bend the incident light as strongly. Knowing the focal length of a lens becomes extremely useful in electron microscopy, because it allows us to accurately predict where the electron beam will focus on a specimen as well as calculate the magnification of the resulting image as it passes through a series of lenses. Various ways for calculating focal length are explained below.

In an optical lens, when incident light is parallel to the face of a converging lens, the beam will converge to one point on the lens' optical axis. The distance between the lens and this point where parallel rays converge is the focal length. Applying parallel rays to a electromagnetic lens results in similar converging behavior, as shown in Figure 3(a). Thus, we can utilize the parallel rays method as one way of calculating focal length.

When the incident light is not parallel, and is instead emitted from a single point, a converging optical lens will bend the rays back inward. Under the right conditions, the rays will converge again at one point on the other side of the lens and form an image of the source point. The locations of the produced image and the original source point are related to each other and the focal length f through the Thin Lens Equation (3)

$$\frac{1}{f} = \frac{1}{d_s} + \frac{1}{d_i} \tag{3}$$

where d_s represents the distance along the optical axis between the source point and the lens, and d_i is the similar distance between the image point and the lens. Sending divergent rays through an electromagnetic lens, as shown in Figure 3(b), results in converging behavior similar to the optical scenario. Thus, we can use the diverging rays method and Equation (3) as another way of determining focal length.

We now have two methods for determining the focal length of a given lens: the straight-forward parallel rays measurement of convergence distance and the divergent rays method of calculating the focal length based upon source and image distances. We utilize both methods in this work primarily to double-check accuracy of our calculations, but also to provide justification for the application of Equation (3) to the domain of electromagnetic lenses, since this equation allows important calculations related to image magnification.

An important difference between calculations in optical and electromagnetic lens domains is that the region over which the optical lens bends light has definite boundaries, while the magnetic field of the electromagnetic lens technically acts over an infinite region with strength falling off according to the inverse cube of axial distance. This significantly impacts the calculation of focal length for the electromagnetic lens because in all the optical calculations rays originate in a region where the lens exerts exactly zero influence. Thus, we need to carefully choose our initial position for the electron beam so that it originates sufficiently far from the lens coil that the initial field strength can be considered negligible and focal length calculations will thus be consistent. Determining a sufficiently accurate initial starting point for the beam will thus be a prerequisite to any actual focal length calculation.

Additionally, we should also observe that convergence can occur on either side of the lens, depending on the strength of the magnetic field and the velocity of the incoming beam. Thus, it is possible to have negative focal lengths (when convergence occurs on the same side of the lens as the beam source) as well as a zero focal length (convergence precisely in the center of the lens).

III. Numerical Simulation

With a set of equations that we cannot solve analytically, we turn to numerical methods to simulate our model. With a working simulation, we can validate it and experiment with the parameters.

A. Implementation of Governing Equations

In order for the numerical simulation of the lens to function, we must calculate the net magnetic field produced by the lens coil at a given point. From Equation (1) and subsequent parametrization, we have a numerically solvable integral expression for such a value. Utilizing a trapezoidal approximation, we can solve this integral within the computational software Matlab.

Additionally, we must also compute the trajectory of each electron through the lens. As discussed previously, we find this by solving the differential equation which results from the Lorentz Force Law and Newton's Second Law of Motion. To make this calculation tractable, we used Matlab's *ode45* variable timestep solver, with the added restriction of a maximum timestep to ensure accuracy.

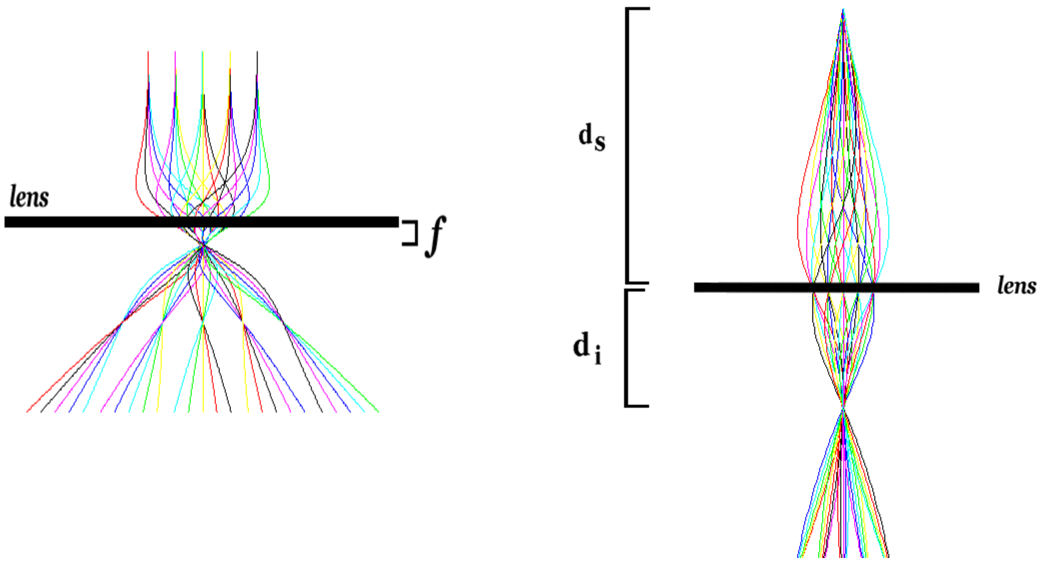
B. Implementation of Electron Projection Methods

With the solver ready, we set up our initial conditions for the two electron projection methods: initially parallel rays and initially diverging rays. Within each method, we fired multiple electrons at the lens, each with unique initial conditions, to ensure greater accuracy.

For the parallel ray method, we used multiple electrons that started on a plane parallel to the coil, at a distance of z_0. Each had the same initial velocity, which was parallel to the z-axis, so that it would strike the lens perpendicularly.

For the diverging ray method, the electrons all started at the same point on the z-axis at a distance of z_0 from the origin. They had the same velocity magnitude, but the direction was changed from electron to electron. The initial distribution of velocities was set up to be evenly spread out and intercept a small circular region on the plane of the coil.

Our preliminary tests revealed success in the focusing process, as seen in Figure 3.



(a) Parallel Rays Method

(b) Diverging Rays Method

Fig. 3. Diagrams of the Parallel Ray (a) and Diverging Ray (b) Methods of Convergence. Electron trajectory paths are multicolored, with each line representing the path of one electron through the lens (shown as a horizontal black line). When electrons start with parallel velocities, as in (a), they converge below the lens at the focal distance, f. In the case of (b), electrons with initially diverging paths will reconvene below the lens according to the Thin Lens Equation (3).

C. *Focal Length Consistency*

As mentioned earlier, before any focal length calculation can be termed accurate, we must determine an initial starting distance z_0 for the beam at which the magnetic field strength is effectively negligible, because we must ensure that the beam passes through the entirety of the lens. In order to determine this starting distance, we conduct a series of tests by firing the same beam through the same lens at various starting positions and determine how far away the beam must originate before the focal length calculation becomes consistent. We plot the calculated focal length as a function of z_0 and look for the trend line to level off at one constant focal length value as z_0 increases. This graph can be seen in Figure 4. This must be done at separately within each electron projection method, and can be further checked by comparing the consistent focal lengths predicted by each method and ensuring that they are the same.

We note from Figure 4 that the parallel rays method will level off to one constant f value when $z_0 >= 1$ m, while divergent rays will level off at approximately the same value only at much greater distances, when $z_0 >= 10$ m.

The observation that focal length consistency for the diverging rays method exists at a z_0 distance on the order of 10^1 meters indicates that the iron shield utilized in real-world lens applications is necessary for an electron microscope to assume a usable scale. Without a shield to

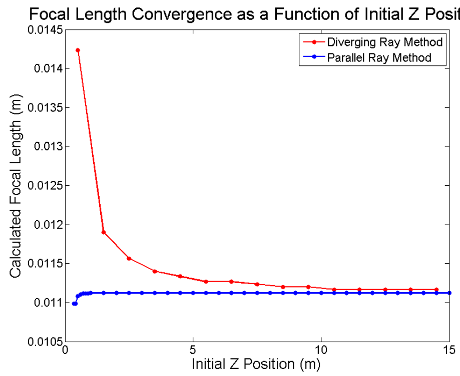


Fig. 4. Focal length convergence as a function of initial z position. Note that the parallel rays method converges to a consistent focal length at much lower initial z value than divergent rays method does. Both level off at approximately the same focal length, verifying the accuracy of each calculation.

contain the magnetic field to a desired region and substantially reduce the distance at which the field becomes negligible, the electron gun which fires the beam at the specimen must be located at an unfeasible distance from the specimen for consistent focusing to occur.

D. Validation

As mentioned previously, no work should be done on the electron by the magnetic field, and thus the electron should gain no kinetic energy throughout its experience in the lens. Our simulation confirms this, calculating the electron's velocity magnitude over time to fluctuate in the worst case by amounts on the order of $10m/s$ for an initial velocity magnitude on the scale of $10^7 m/s$. We can regard this small fluctuation as inevitable yet negligible numerical error and thus consider our model to accurately conserve energy.

Additionally, we were able to effectively use our focal length calculations in reverse to verify their success. Using the divergent rays method, we shot electrons at the lens from a z_0 equal to the average focal length for that lens and observed as expected that the electrons emerged from the other side of the lens with parallel trajectories. This behavior confirms that our simplified lens can indeed be considered to have a focal length in the same sense as an optical lens, and also that our calculation of focal length is indeed correct.

IV. Analysis

A. Methodology

With our simulated model operational and validated, we used it to observe the dependency of a lens' focal length on specific parameters. Specifically, our formulation of the model indicates that the variables of R, I, and $|\vec{v}|$ are the only definable parameters which play a role in influencing the focal power of a given lens. R and I are the only definable factors which influence the power of the magnetic field, and thus the strength of the lens, while $|\vec{v}|$ plays a role in determining the force an electron feels due to the field, and thus its tendency to converge. We thus set about conducting systematic tests concentrating on each of these three variables: coil radius length, current magnitude in the lens wire, and initial electron velocity magnitude. We examined each over a wide variety of values while holding the other two constant, then plotted the observed trend in focal length and attempted to express mathematically the dependence of focal length on the variable in question.

V. Radius Length

A plot of average focal length as a function of radius length appears in Figure 5. Both projection methods were used in this calculation.

We see that the relationship between f and R is almost perfectly linear. More importantly, they appear to be directly related, as an extension of this trend line will cross the origin. We thus express this relationship as

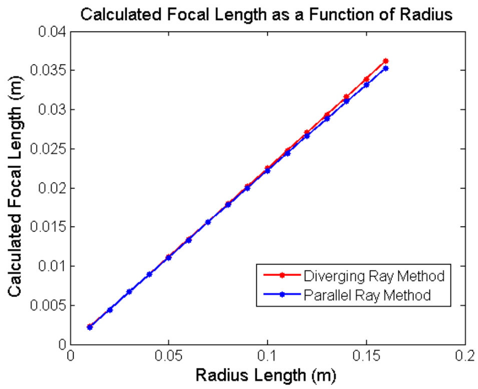


Fig. 5. Average focal length as a function of radius length. A clear linear trend exists. A high correlation exists between both calculation methods.

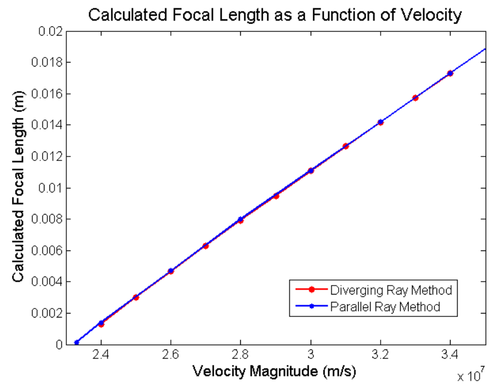


Fig. 6. Average focal length as a function of velocity magnitude. A clear linear trend exists. A high correlation exists between both calculation methods. Also note that the focal length reaches zero at a specific velocity value.

$$f \propto R \tag{4}$$

A direct relationship between focal length and radius length makes sense, because a larger radius will cause the field to be weaker near the center of the lens. As a result of this weaker field, incident electrons will feel a weaker force pushing them toward the optical axis as they pass through the lens. Thus, a lens with larger radius will be less powerful and will cause convergence at larger focal lengths.

VI. Velocity Magnitude

A plot of average focal length as a function of the electron's initial velocity magnitude appears in Figure 6. Again, both projection methods were used in this calculation.

We see that the relationship between f and $|\vec{v}|$ is almost perfectly linear. However, there exists some velocity magnitude v_0 at which the focal length is zero. Taking this into account, we express this relationship as

$$f \propto (v - v_0) \tag{5}$$

The linear relationship between focal length and velocity makes qualitative sense because as a particle's velocity increases, it will move through the lens with a greater $\hat{z}$ component of velocity, and will thus travel more distance along the z-axis before converging. Similarly, as particle velocity decreases, it will travel less distance before converging. An important theoretical observation to make is that electrons can converge on either side of the lens if the electrons are sufficiently slow or the field is sufficiently strong, indicating that the presence of a zero crossing point v_0 in this relationship is also necessary. Further investigation yielded that the value of v_0 was directly related to the current applied to the lens and was independent of radius length. We can express the zero crossing velocity as $v_0 = kI$. Thus, the relationship between velocity and focal length, considering the zero crossing's dependence upon current, can be formalized as

$$f \propto (v - kI) \tag{6}$$

The value of k was consistently found to be approximately $2.321 * 10^4 \frac{m}{As}$.

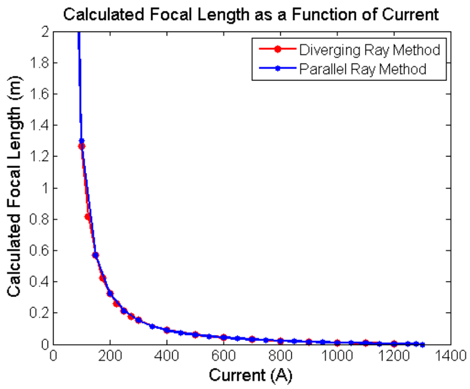


Fig. 7. Average focal length as a function of current. The trend appears to follow an inverse square relationship or similar function. A high degree of correlation exists between both calculation methods.

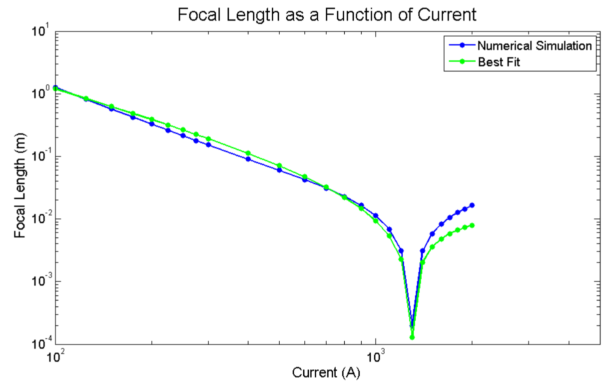


Fig. 8. Comparison of the absolute value of numerically calculated focal length as a function of current (in blue) and the absolute value of the proposed line of best fit (in green) on a log-log scale. Note that the fit line gives reasonably accurate results but by no means provides perfect fit.

VII. Current

A plot of average focal length as a function of the current running through the coil appears in Figure 7. Again, both projection methods were used in this calculation.

This inversely decaying relationship between current and focal length makes qualitative sense because magnetic field strength is directly proportional to current. It follows that if current increases, magnetic field strength increases, the lens gets stronger, and electrons converge closer to the lens, yielding lower focal length. Thus, as current increases, focal length decreases.

Determining a mathematical relationship between current and focal length, however, proved to be difficult. The data appears to follow a $f \propto \frac{1}{I^n}$ relationship with $1 <= n <= 2$. Additionally, we accounted for current's role in determining the zero crossing of focal length. The best fitting curve was found to follow the equation

$$f \propto \frac{v - kI}{I^{1.5}} \tag{7}$$

This best fit line can be compared against the numerical calculation in Figure 8. It gives a fairly uniform fit but is far from perfect. It matches much better for lower current values than higher ones.

VIII. Conclusion

We succeeded in constructing a model of a simple electromagnetic lens and in determining some basic mathematical trends of focal length dependence. This understanding is key to developing a comprehensive knowledge of how an electron microscope works. Through the use of this model, we begin to quantitatively understand the focusing process in electron microscopy, which has broad applications in fields like biology and material science.

The limitations of this work are numerous. First of all, our model dealt with only one electron passing through the lens at a time, which is highly idealized and would need to be altered to more accurately model the real lens system, since the charged nature of electrons would surely result in some repellant electrostatic forces within the beam which could alter focusing ability in the lens. Additionally, our assumption of infinitely thin wire is certainly unrealistic, especially when carrying currents of the magnitudes we used in our simulation.

Also, our model does not account for relativistic effects in the electrons, which in the simulation are moving at speeds near enough to the speed of light ($|\vec{v}| \approx \frac{c}{10}$) to possibly merit relativistic consideration. Finally, numerical error certainly accumulated from our magnetic field calculation, our trajectory calculation, and our focal length calculation, and must be considered as a limiting factor to our accuracy.

This work has opened up many possibilities for further research. First, finding a more precise mathematical relationship between current and focal length could be highly advantageous, since it would ideally allow us to develop an equation for focal length in terms of all three parameters. Also, developing a mathematical model of the magnetic field generated by the iron shroud, as used in real electron microscopes, would be a welcome improvement in our work's real-world accuracy and relevance. Examining the effects of multiple electromagnetic lenses in series would be an important step to understanding how image magnification happens within electron microscopes. Additionally, the development of a formalized, analytical proof which explains why all electron paths must converge at one point in an electromagnetic lens would be highly useful, both for providing a more robust theoretical basis for this work and for its potential for more accurately elucidating the dependence of focal length on various lens parameters.

This work hopefully provides grounding for a basic understanding of how focusing works in an electromagnetic lens, but there certainly exist many avenues for improvement and advancement.

Acknowledgment

The authors would like to thank Dr. John Geddes and Dr. Mark Somerville for their assistance and direction in the preliminary stages of this project.

References

[1] Goohew, Peter J. et. al., *Electron Microscopy and Analysis*, 3rd ed. London: Taylor and Francis, 2000.