Optimizing the Human Dynamics of the Caber Toss

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Abstract

Caber tossing is a Scottish athletic event in which participants flip a telephone-pole sized log in the air. For a throw to be successful, the caber must fully flip end over end. In this paper, we develop a mathematical model of the event and investigate the optimization of human parameters (running acceleration and lifting force) that yield a successful toss with minimal energy expenditure. Our findings suggest that higher initial running accelerations reduce the total energy necessary to successfully flip a caber.

1 Introduction

The caber toss is a traditional Scottish Highland game involving a wooden pole around 5 to 6 meters long called a caber. In the game, participants attempt to lift this approximately 50 kilogram object, balance it vertically on one end, and toss the pole forward so that it flips end over end before landing on the ground. A photograph of the lifting portion of the caber toss can be seen in Figure 1. The participant often gains a running start before halting suddenly and exerting the lifting force. The object of the game is not to achieve distance, but to cause the caber to flip over about 180° and land so that it points directly away from the tosser. A toss is scored based on how far the caber's resulting angle at rest deviates from pointing directly away from the tosser. A diagram illustrating the success or failure possibilities in two-dimensions appears in Figure 2.

In this paper, we examine the human factors that contribute to the success and energy expenditure of a caber toss. To do this, we begin by developing a simplified working model of the behavior of a caber under certain initial conditions; this is derived through analysis of the forces and torques acting on the

Figure 1: A photo of a caber toss. Notice the angle the caber makes with the ground as the tosser begins to exert an effort. This angle proves to be a key factor in the success or failure of a lift, as will be demonstrated later. Photo courtesy Cambridge (Ontario) Highland Games.

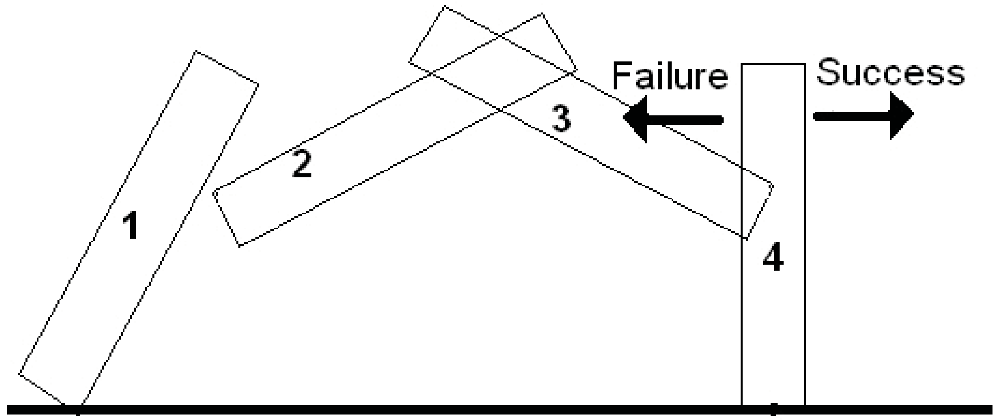

Figure 2: The temporal progression of the caber toss. We see each phase marked with its appropriate number. Note that in Phase 4 the caber may either fall backwards, causing failure, or forwards, resulting in success.

caber throughout the toss. Using the model, we can observe the success or failure of a toss in relation to controllable human parameters, namely running acceleration and lifting force.

We operate under several assumptions that allow us to model the system with greater ease. First, the caber is treated as a line rather than a cylinder. We also ignore friction and air resistance. Most importantly, the problem is treated two-dimensionally, meaning that the landing angle is a non-issue. Essentially, we are concerned with the success or failure of a toss, and in the case of success, we investigate the work expended to achieve such a toss.

2 Theory

To create a reasonable model, we must consider both the rotational and translational motion of a rigid body. The governing equations for such motion can be derived by examining the force and momentum experienced by the caber. In terms of linear motion, we use Newton's second law expressed in terms of momentum:

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{1}$$

where $\vec{F}$ is the force and $\vec{p}$ is the linear momentum of the caber. Similarly, we can express this in terms of rotational motion:

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{2}$$

where $\vec{\tau}$ is torque and $\vec{L}$ is angular momentum. Furthermore, linear momentum is expressed as $\vec{p} = m\frac{d\vec{r}}{dt}$, and angular momentum is expressed as $\vec{L} = \vec{r} \times \vec{p}$. By equating these expressions of momentum and force, we can find governing equations that describe both linear and rotational acceleration for the caber throughout its range of motion. Knowing the expression for acceleration, we can integrate the expressions numerically to find a time-based function of position, thereby allowing us to plot the caber's trajectory.

3 Modeling

The first step toward mathematical analysis of the caber toss is dividing the motion into distinct phases based upon the forces that act on the caber. For simplicity, we assume our model begins when the tosser has successfully picked up the caber and balanced it in his hands. In the first part of our formulation, the

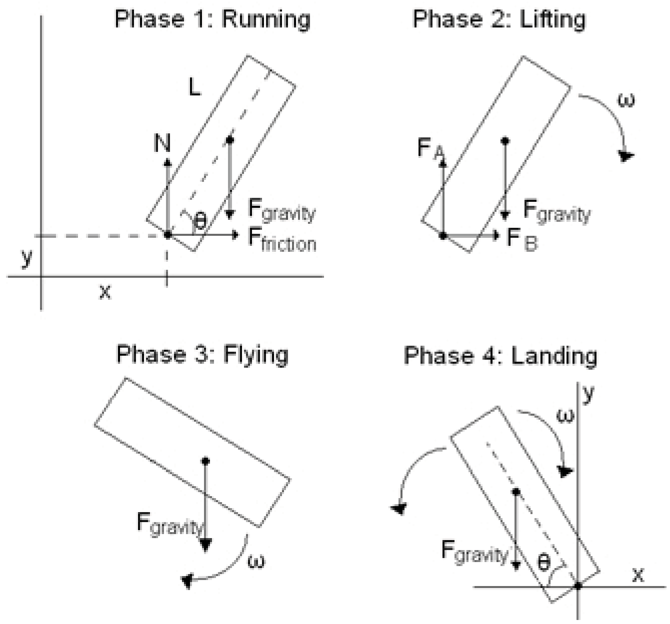

Figure 3: Free body force diagrams of distinct phases of motion. Phases 1, 2, and 3 utilize the coordinate system drawn explicitly in Phase 1. Phase 4 necessitates a new system with the origin at the contact point, making calculations understandable and efficient. Angular velocity of the caber is shown as ω. Note that in Phase 4, the caber may fall either forward or backward as it lands, thus the two seemingly opposing lines of motion. L is the length of the caber in all cases.

tosser begins to run forward, accelerating the caber horizontally. Next, the tosser halts his own motion and exerts a vertical force on the caber by lifting. When he releases the caber, it flies through the air affected only by gravity. Finally, the caber lands on one end, which becomes a fixed pivot, and the other falls to the ground due to gravity. These phases and their respective forces are shown in Figure 3. Below, we examine the detailed assumptions, transition conditions, and governing equations associated with each phase. We set the length of the caber l at 5 meters, and the mass of the caber m at 50 kilograms.

Phase 1: Running While running, the tosser applies a constant horizontal force to accelerate the caber forward, expressed as an acceleration $\ddot{x}$. The angular acceleration of the caber $\ddot{\theta}$ is given in terms of the tosser's horizontal acceleration $\ddot{x}$ as follows

$$\ddot{\theta} = \frac{3}{2l}(\ddot{x}\sin(\theta) - g\cos(\theta)) \tag{3}$$

During this phase there is no acceleration in the vertical direction ($\ddot{y} = 0$), because the runner is essentially holding the caber in place. Although this equation governs the caber's angular acceleration, $\ddot{\theta}$ is essentially zero throughout the running phase because we set the caber's initial angle θ_0 to a value such that the torque due to gravity exactly cancels the torque due to the running horizontal force. We do this so that the caber remains fixed at one specific angle, which is approximately what we observed in our study of caber toss video footage. We allow the running phase to last until the tosser has traveled a fixed distance d horizontally, at which point we transition to the lifting phase. We estimate d to be 7 meters in this work, a value roughly consistent with observed video footage of the event.

Phase 2: Lifting After the tosser has run through the given distance d, we transition to the lifting phase. The caber continues to travel horizontally at the velocity with which it left the running phase. However, a vertical lifting force A and a horizontal stabilizing force B are now applied to the caber, yielding the following governing equations

$$\ddot{\theta} = \frac{\frac{A}{m}\cos(\theta) + \frac{l}{2}\dot{\theta}^2\sin(\theta)\cos(\theta)}{\frac{l}{2}\cos^2(\theta) - \frac{2l}{3}} \tag{4}$$

$$\ddot{y} = \frac{A}{m} - g - \frac{l}{2}\ddot{\theta}\cos(\theta) + \frac{l}{2}\dot{\theta}^2\sin(\theta) \tag{5}$$

$$\ddot{x} = \frac{B}{m} + \frac{l}{2}\ddot{\theta}\sin(\theta) + \frac{l}{2}\dot{\theta}^2\sin(\theta) \tag{6}$$

During the lifting, linear acceleration is constrained to the vertical direction. As mentioned earlier, the supported end of the caber continues to move horizontally at the same velocity that was reached at the end of the running phase. However, by lifting up on one end of the caber, the resulting rotational motion would normally cause the supported end to accelerate horizontally. To correct this, a stabilizing horizontal force B is posed during lifting at a magnitude precisely large enough to balance the horizontal acceleration caused by rotation. This results in a condition where $\ddot{x} = 0$. We also note that the angular acceleration is ultimately dependent only on the vertically applied force, the caber's angle, and its angular velocity.

We assume that as lifting occurs, the tosser's own body remains fixed in one position, no longer moving horizontally. However, as mentioned above, the caber continues moving horizontally at a constant velocity. We thus reason that lifting can only continue as long as the caber remains within the reach of the tosser's arms. We set the horizontal reach at .5 meters and the vertical reach at .75 meters, to approximate the limits of the average human body. After the caber exceeds either of these limits, no more lifting force is applied and we transition to the flying phase.

It is important to note that this lifting model is not fully accurate; in reality, a caber tosser actually applies a horizontal force that prevents the supported end of the caber from moving at all in the horizontal direction (i.e. $\dot{x} = 0$ *and* $\ddot{x} = 0$). So, all of the caber end's upward motion occurs along a stationary axis. By essentially stopping the caber in its tracks, the linear momentum from the running phase is translated completely into angular momentum, thereby significantly increasing the caber's angular velocity. We decided to use the simplified lifting model described above for ease of manipulation. This choice requires much higher lifting forces for a successful toss, but it does not change the qualitative behavior of the caber because the forces acting on the system are unchanged.

Phase 3: Flying After the caber leaves the tosser's grasp, it flies through the air affected only by gravity. The governing equations of this regime follow

$$\ddot{y} = -g \tag{7}$$

$$\ddot{\theta} = 0 \tag{8}$$

During the flying phase, there is no angular acceleration because the only force acting on the caber is gravity, which causes no torque. Thus, the caber continues moving horizontally forward and spinning at the angular velocity reached at the end of the lifting phase while accelerating downward at a rate equal to the gravitational constant. The trajectory of the caber in this phase follows the classical parabolic shape of projectile physics. We transition out of this phase when either end of the caber strikes the ground ($y = 0$).

Phase 4: Landing After striking the ground with one end, we immediately consider that end a fixed pivot, and, for simplicity, do not consider any effects due to impact with the ground. We assume that the angular velocity of the caber through the flying phase remains the initial angular velocity in this phase. Depending on the angle at which the caber strikes the ground and its angular velocity at this point, the caber will fall forwards or backwards until it lies flat on the ground. The governing equation for the caber's angle to the horizontal in this regime follows.

$$\ddot{\theta} = \frac{-3g\cos(\theta)}{2l} \tag{9}$$

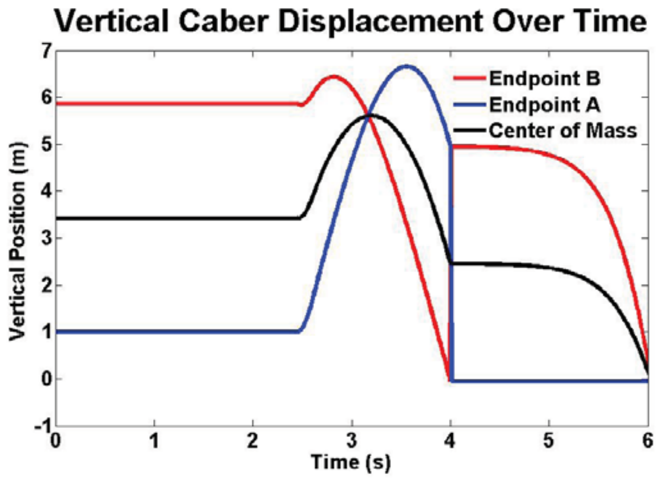

Figure 4: Vertical caber displacement over time for each endpoint and the center of mass. The caber's height remains constant while running, from t = 0 until around t = 2.5 seconds. While lifting, the caber begins to both rotate and rise vertically. After the lifting phase, we notice the parabolic shape of the center of mass' trajectory, as expected. Shortly after 3 seconds, the caber becomes fully horizontal (all lines intersect) before endpoint A becomes higher than endpoint B. The discontinuity around time = 4 s is due to the model's need to switch coordinates systems when one end strikes the ground, as discussed in Figure 3. The ends simply switch values at this point, and shortly after this we see and the other end and the center of mass fall to the ground (y = 0).

It makes sense that the angular acceleration at a given moment should depend solely on its current angle and the gravitational constant. We can assert from this equation that if the caber lands completely vertically (and with no angular velocity), it will not fall over. This situation would never happen in a real toss, since angular velocity will always be nonzero, but it is useful for qualitatively understanding the $\ddot{\theta}$ governing equation. With the landing phase successfully modeled, we transition out of this phase and exit our simulation when the caber lies flat with both ends on the ground ($y = 0$).

4 Analysis and Optimization

Qualitatively, we may observe that the caber behaves in a reasonable manner. In Figure 4, a plot of the caber's flight path over time, we see the ends of the caber flip over one another, making a transition at approximately $t = 3s$, when the ends are at the same height; ultimately, the caber lands horizontally, so the ends are both at a height of 0 m. Additionally, simple test cases indicate that if the lifting force is zero, the caber indeed falls straight to the ground as expected, and if the lifting force is exactly equal to the caber's weight force, the caber does not move vertically. For a more rigorous, quantitative verification, we conducted energy conservation analysis of all phases. In both running and lifting phases, since the tosser adds energy into the system by doing work on it, we calculated the work done in each phase and compared it to the increase in energy experienced in each phase. These two quantities were found to be equal, as expected by the work-energy principle. In the flying and landing phases, total system energy should be conserved, since there is no external force acting on the system, and it was.

Having developed a working model, we can vary the parameters of initial running acceleration ($\ddot{x}$) and vertical lifting force (B) in order to change the energy required to complete a toss. Based on the running capabilities of an average human, we chose to investigate $\ddot{x}$ values between 2 and 3.5 m/s^2, which led to lifting forces between 8000 and $10,000\ N$. As we can see in Figure 4, there are some combinations of $\ddot{x}$ and B that do not yield a successful toss. These unsuccessful tosses are indicated by the dark red zone. However, among successful tosses, we note that the human lifting forces required are between $8,000$ and $10,000\ N$, which are quite unrealistic values; a more reasonable lifting force a human being could exert would be in the range of $1000 - 2000$ N. We attribute this to the simplifications made in our lifting model, as discussed in **Modeling**. However, despite the inaccuracy of the values themselves, we can observe a general qualitative

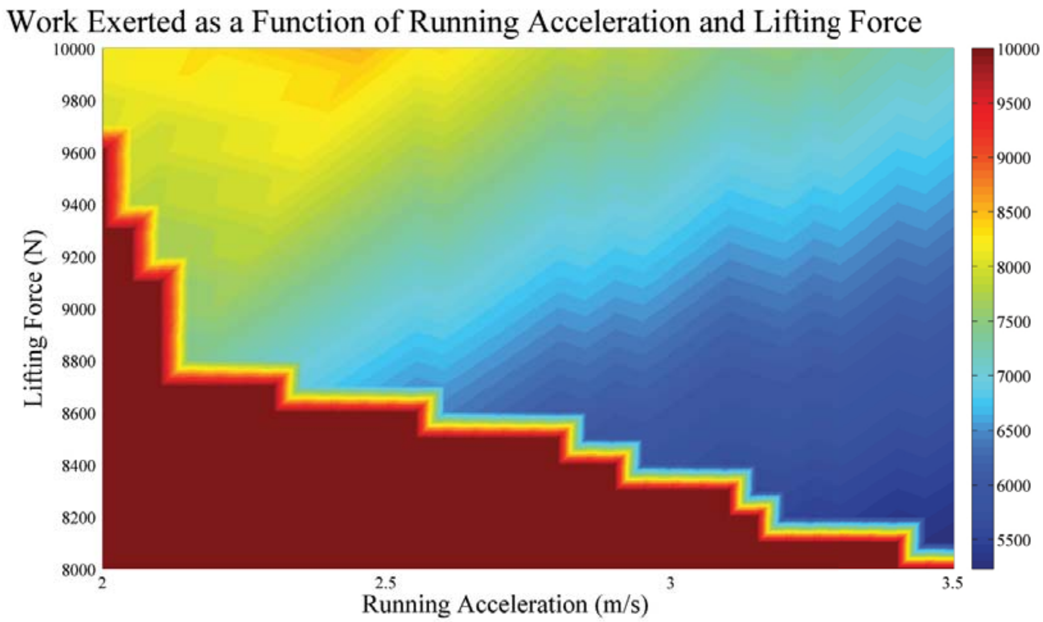

Figure 5: Contour plot of work exerted for successful tosses as a function of initial running acceleration and vertical lifting force. The dark maroon area in the lower left represents a region of values which result in a failed toss. The gradient region represents a combination of values which result in a successful toss. The colorbar to the right of the figure represents the amount of work exerted to complete the toss. High values of work (on the order of 9000 J) are shaded orange-yellow and low values (on the order of 6000 J) are colored blue, with a smooth gradient in between. We see that as running acceleration increases, the amount of lifting force necessary to complete a successfully toss decreases, as does the amount of work. Work here was computed as the net energy gained by the system from the initial to final state.

trend in the data: as running acceleration $\ddot{x}$ increases, the minimum lifting force necessary for a successful toss decreases.

To understand this, we must revert back to the running phase; recall that during this phase, the starting angle is matched exactly to the runner's acceleration so that the caber's angle does not change while in motion. As the tosser's $\ddot{x}$ value increases, the angle with respect to the ground at which he is able to support the caber decreases. This closer proximity to the ground yields two advantages. First, and most obviously, if we want the caber to flip end over end before landing, any decrease in the initial angle from the ground is beneficial, since this reduces the necessary total angular displacement. More significantly, though, a closer angle to the ground also gives an advantage in torque application. When we enter the lifting phase with a smaller θ, the upward applied force has a greater lever arm since the center of mass is farther forward; thus, the caber experiences more torque and consequently undergoes a greater increase in angular momentum. As such, the caber is more likely to continue falling forward after landing. This effect reaches a maximum when the caber is parallel with the ground, because at that angle, all of the lifting force is perpendicular to the lever arm. However, it is also important to keep in mind that the force during the lifting phase should not just apply torque, but must also impart some vertical linear velocity to the center of mass. Otherwise, the caber will hit the ground before it can complete a rotation. This velocity depends on the portion of the lifting force directed through the caber's center of mass. Thus, the caber's initial angle, which is uniquely associated with a particular running acceleration, must balance these two criteria, with an optimal value probably somewhere in the neighborhood of $\theta = 45°$. The reason we do not see a clear optimal running acceleration on Figure 4 is because we chose to restrain ourselves to realistic values for $\ddot{x}$, which lead to initial angle values higher than 45°.

We can conclude our analysis with the general knowledge that raising the tosser's running acceleration will substantially increase the likelihood of a successful toss as well as reduce the amount of energy or work he or she must exert. This is primarily due to the fact that at higher running accelerations the caber can be held stably while running at angles closer to the ground, making flipping the caber over an easier task.

5 Limitations

Model Inadequacy As mentioned earlier, our simplified model does not perfectly reflect the actual dynamics of the caber toss. First, by focusing upon two-dimensions, we do not adequately model the process of applying force with two arms while lifting, which is a major component of the actual event. In reality, a tosser is essentially graded based upon how evenly he or she can apply the lifting force between both arms. The toss is ultimately assessed by how close the caber's final angle on the plane of the ground is to the line of his running path. The two-dimensional simplification was necessary to make the problem tractable, but severely limits our ability to reflect reality.

Another major problem within our model was our failure to account for changes in momentum between phases. As discussed earlier, this resulted in unrealistic force values in the lifting phase. Correcting for this momentum transfer would most likely allow the values of the numerical simulation of the model to approach real-world values, especially for the lifting force and the work done. However, our conclusion about the value of the running phase still holds, because the angle between the caber and the ground would still be smaller if the running acceleration was higher, which is beneficial.

We also fail to account for a momentum transfer when the caber strikes the ground at landing. In our model, the angular velocity was kept constant between the flying and landing phases. This simplification affects the accuracy of our numeric results, but the general relationships between running acceleration, lifting force, success, and work expenditure should remain the same.

6 Conclusion

We have determined that for a successful caber toss, the minimum work expended decreases with increasing running acceleration within realistic human values. This suggests that an aspiring caber tosser should train for speed more than for brute strength, because this will ultimately have a greater impact on the success or failure of a toss. Our model of the caber toss is by no means exhaustive or complete. Most notable among its inaccuracies is the flawed lifting model. There remain many further directions for enhancement. A few suggestions appear below:

- Accurately represent the momentum transfer between phases
- Account for the effect of air resistance upon the caber
- Allow for the caber's angle to fall slightly while running
- Model the toss in three dimensions and add a second lift arm

This work is hopefully just the beginning of a grand tradition of mathematically modeling and optimizing the great sport of caber tossing.

Acknowledgements

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Biography

Giulia Fanti and Michael Hughes are proud members of Olin College of Engineering's Class of 2010. Both share a passion for caber tossing and believe the sport to be the most divine and pure of any athletic event. In the future, they aspire to new tossing heights and hope to become co-leaders of the noble Clan McTavish. Their favorite foods are haggis and rumbledethumps.